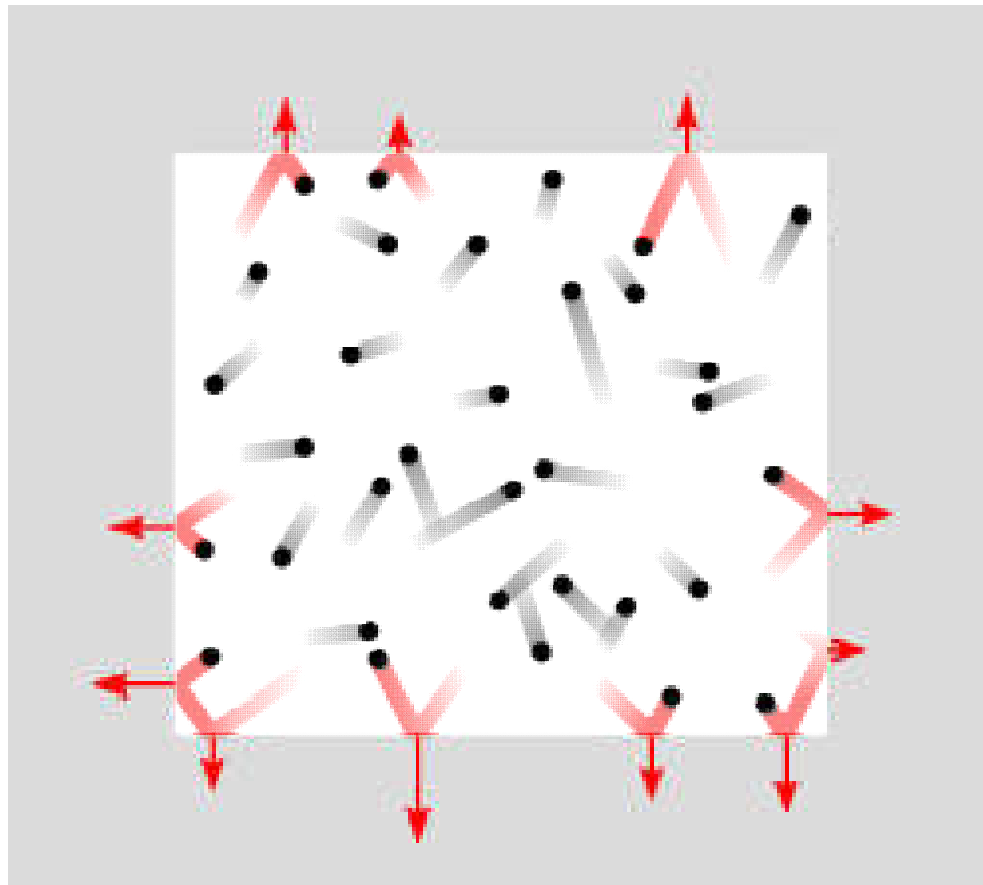


# Fluid statics

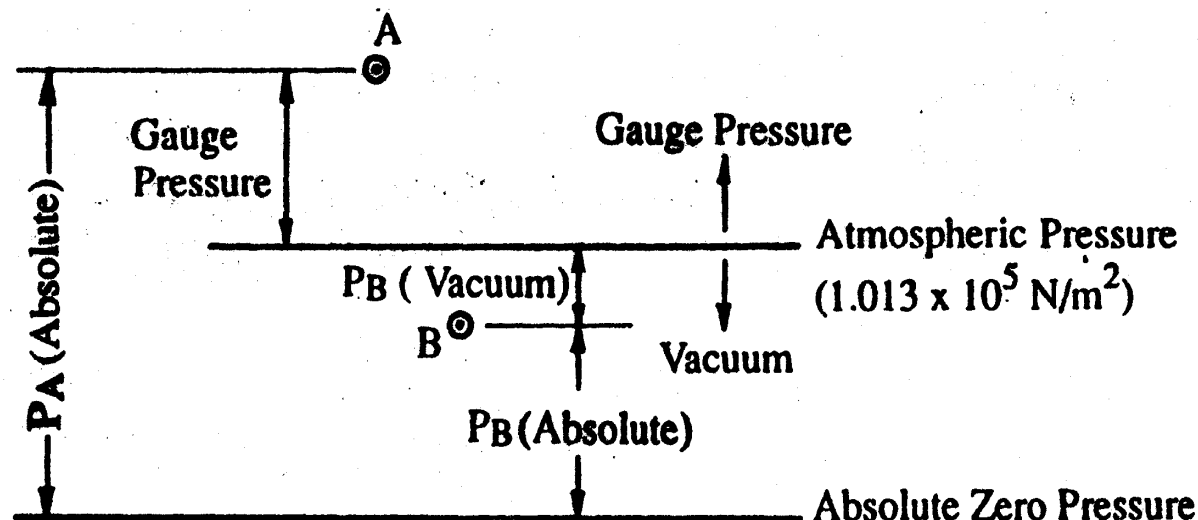


## Pressure, $p$

The pressure at a point in a fluid continuum is defined as *the normal compressive force per unit area at that point*. In other words, the compressive normal stress is referred to as the pressure; It is actually the measure of momentum exchange between colliding molecules.

$$P = -\sigma$$

It may be noted that a *fluid can withstand high compressive stresses or pressure but it is weak in tension*. Atmospheric pressure may also serve as a suitable reference for pressure measurement. Pressure above the atmospheric value is called gauge pressure and that below it is called vacuum or sub-atmospheric pressure.



$$P_{\text{gage}} = P_{\text{abs}} - P_{\text{atm}}$$
$$P_{\text{vac}} = P_{\text{atm}} - P_{\text{abs}}$$

Fig.1. Pressure Reference

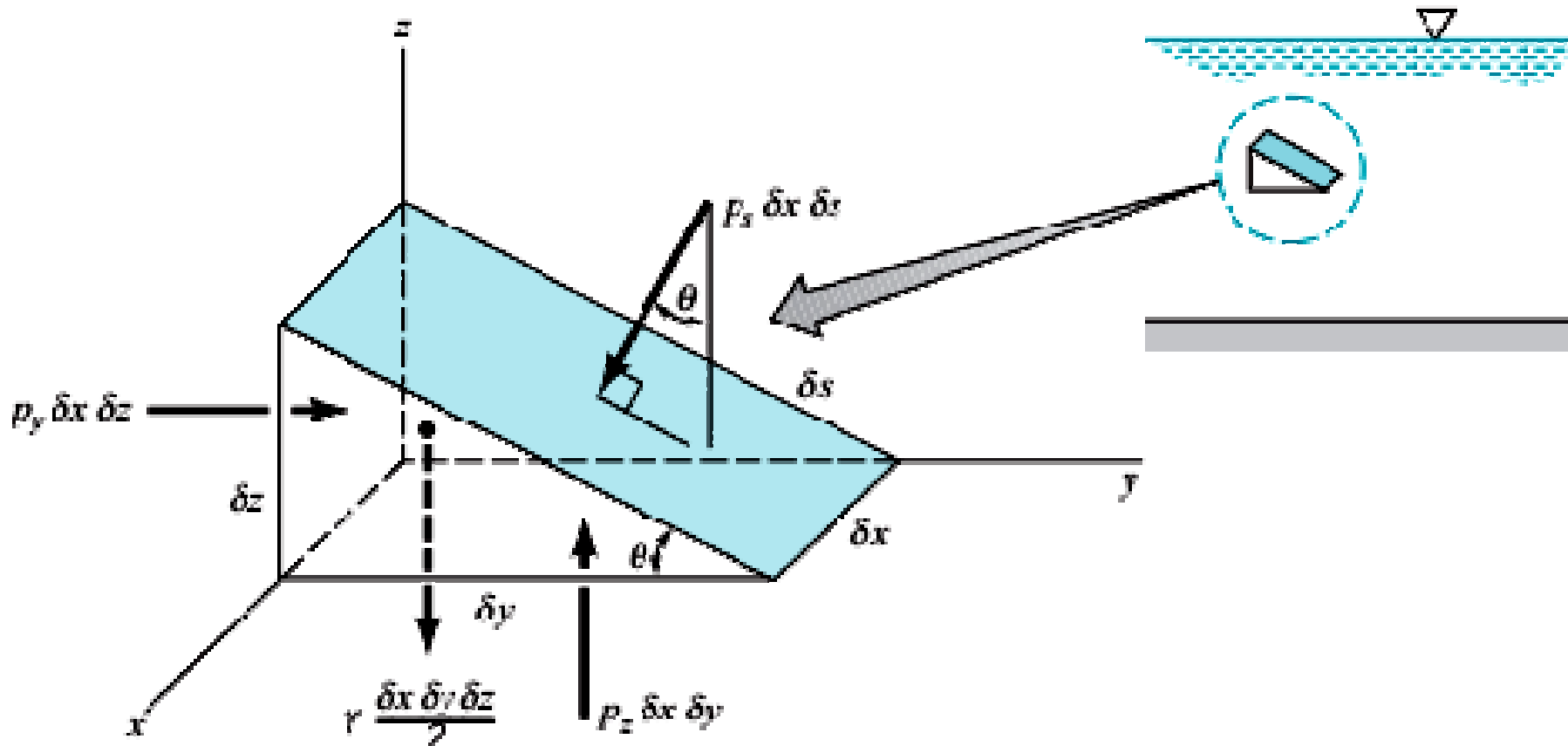
In fluid statics *there will be no relative motion between the adjacent fluid layers*. As a result there will be *no velocity gradient and the shear stress will be zero*. So the viscosity of a fluid has no effect on fluids at rest and therefore, *the ideal and real fluids will behave in the same way*. Here the forces will be due to fluid pressure and gravity. As there is no shear stress, the frictional force is zero.

## Pressure at a Point

In fluids under static conditions pressure is found to be independent of the orientation of the area. This concept is explained by **Pascal's law** which states that *the pressure at a point in a fluid at rest is equal in magnitude in all directions*.

Consider a wedge shaped element in a volume of fluid as shown in Fig.2. Since we are considering the situation in which there are no shearing stresses, the only external forces acting on the wedge are due to the pressure and the weight. For simplicity the forces in the  $x$  direction are not shown, and the  $z$  axis is taken as the vertical axis so the weight acts in the negative  $z$  direction. Although we are primarily interested in fluids at rest,

to make the analysis as general as possible, we will allow the fluid element to have accelerated motion. The assumption of zero shearing stresses will still be valid so long as the fluid element moves as a rigid body; that is, there is no relative motion between adjacent elements.



**Fig.2:** Forces on an arbitrary wedge-shaped element of fluid.

The equations of motion (Newton's second law,  $\mathbf{F} = m\mathbf{a}$ ) in the  $y$  and  $z$  directions are, respectively

$$\sum F_y = p_y \delta x \delta z - p_s \delta x \delta s \sin \theta = \rho \frac{\delta x \delta y \delta z}{2} a_y \quad [m = \rho V]$$

$$\sum F_z = p_z \delta x \delta y - p_s \delta x \delta s \cos \theta - \gamma \frac{\delta x \delta y \delta z}{2} = \rho \frac{\delta x \delta y \delta z}{2} a_z \quad [\gamma = \rho g]$$

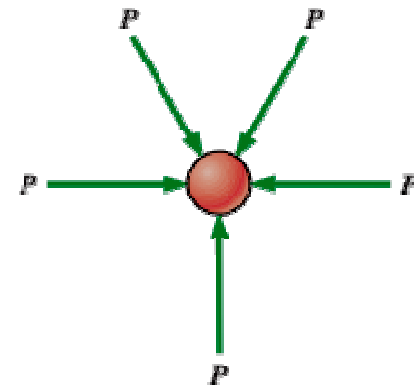
where  $p_s, p_y$  and  $p_z$  are the average pressures on the faces,  $\gamma$  and  $\rho$  are the fluid specific weight and density, respectively, and  $a_y, a_z$  the accelerations. Note that a pressure must be multiplied by an appropriate area to obtain the force generated by the pressure. It follows from the geometry that

$$\delta y = \delta s \cos \theta \quad \delta z = \delta s \sin \theta$$

$$p_y - p_s = \rho a_y \frac{\delta y}{2}$$

so that the equations of motion can be rewritten as

$$p_z - p_s = (\rho a_z + \gamma) \frac{\delta z}{2}$$



Since we are really interested in what is happening at a point, *we take the limit as  $\delta x$ ,  $\delta y$ , and  $\delta z$  approach zero (while maintaining the angle  $\theta$ )*, and it follows that

$$p_y = p_s \quad p_z = p_s$$

$$\text{Or, } p_s = p_y = p_z$$

The angle was arbitrarily chosen so we can conclude that *the pressure at a point in a fluid at rest, or in motion, is independent of direction as long as there are no shearing stresses present.* This important result is known as ***Pascal's law***, named in honor of **Blaise Pascal**.

### **Pascal's Law is valid**

- (a) *for a fluid at rest, whether the fluid is viscous or inviscid, compressible or incompressible.*
- (b) *for a liquid in solid-body movement, either at constant linear acceleration or at constant rotational velocity in a container; here again there is no relative motion between different fluid layers and*
- (c) *for the flow of an inviscid fluid and for ideal fluid flow where  $\mu = 0$  corresponds to absence of shear stresses.*

For moving fluids in which there is relative motion between particles (so that shearing stresses develop), the normal stress at a point, which corresponds to pressure in fluids at rest, is not necessarily the same in all directions. In such cases the **pressure is defined as the average of any three mutually perpendicular normal stresses at the point.**

$$p = -\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$$

## Units Of Pressure

$$1 \text{ Pa} = 1 \text{ N/m}^2$$

$$1 \text{ bar} = 10^5 \text{ Pa} = 0.1 \text{ MPa} = 100 \text{ kPa}$$

$$1 \text{ atm} = 101,325 \text{ Pa} = 101.325 \text{ kPa} = 1.01325 \text{ bars}$$

$$1 \text{ kgf/cm}^2 = 9.807 \text{ N/cm}^2 = 9.807 \times 10^4 \text{ N/m}^2 = 9.807 \times 10^4 \text{ Pa}$$

$$= 0.9807 \text{ bar}$$

$$= 0.9679 \text{ atm}$$

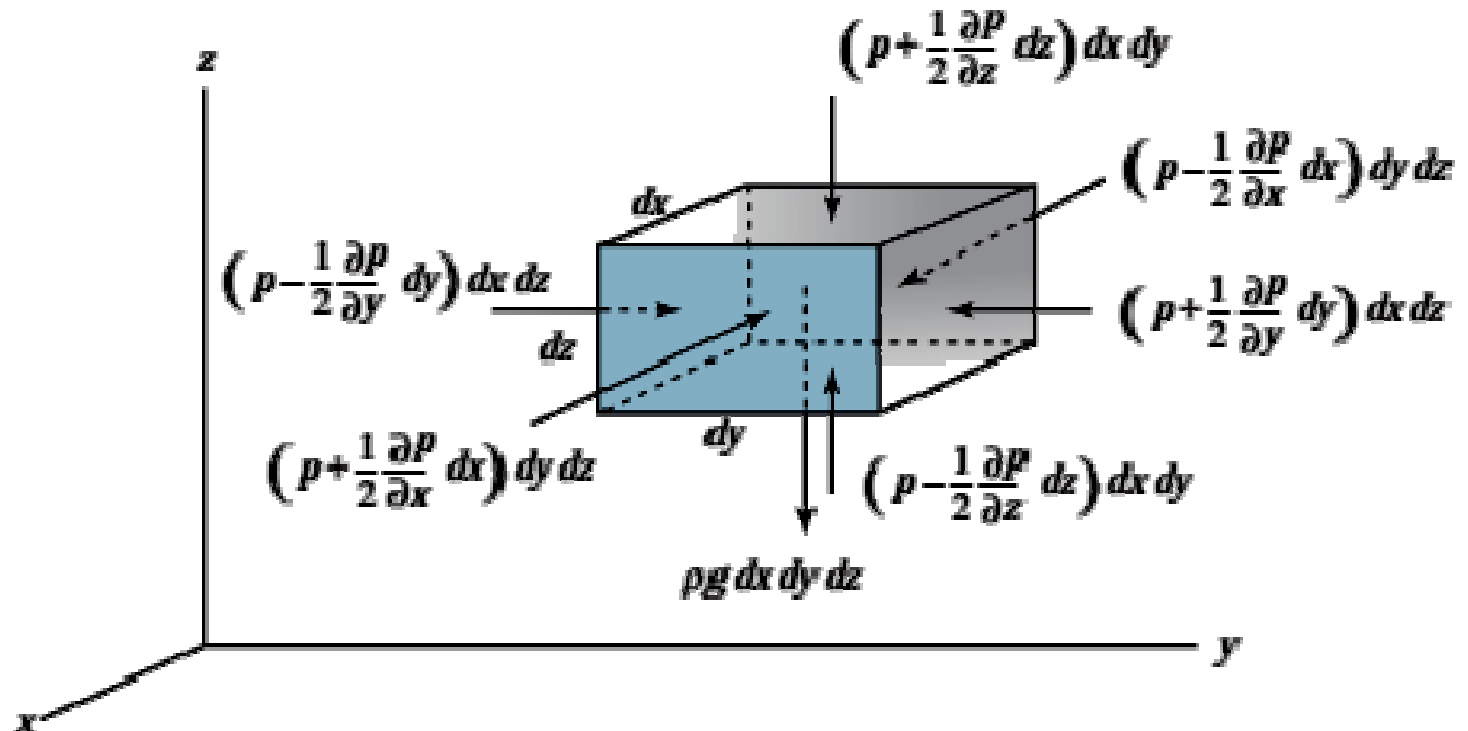
$$1 \text{ kgf/cm}^2 = 14.223 \text{ psi}$$

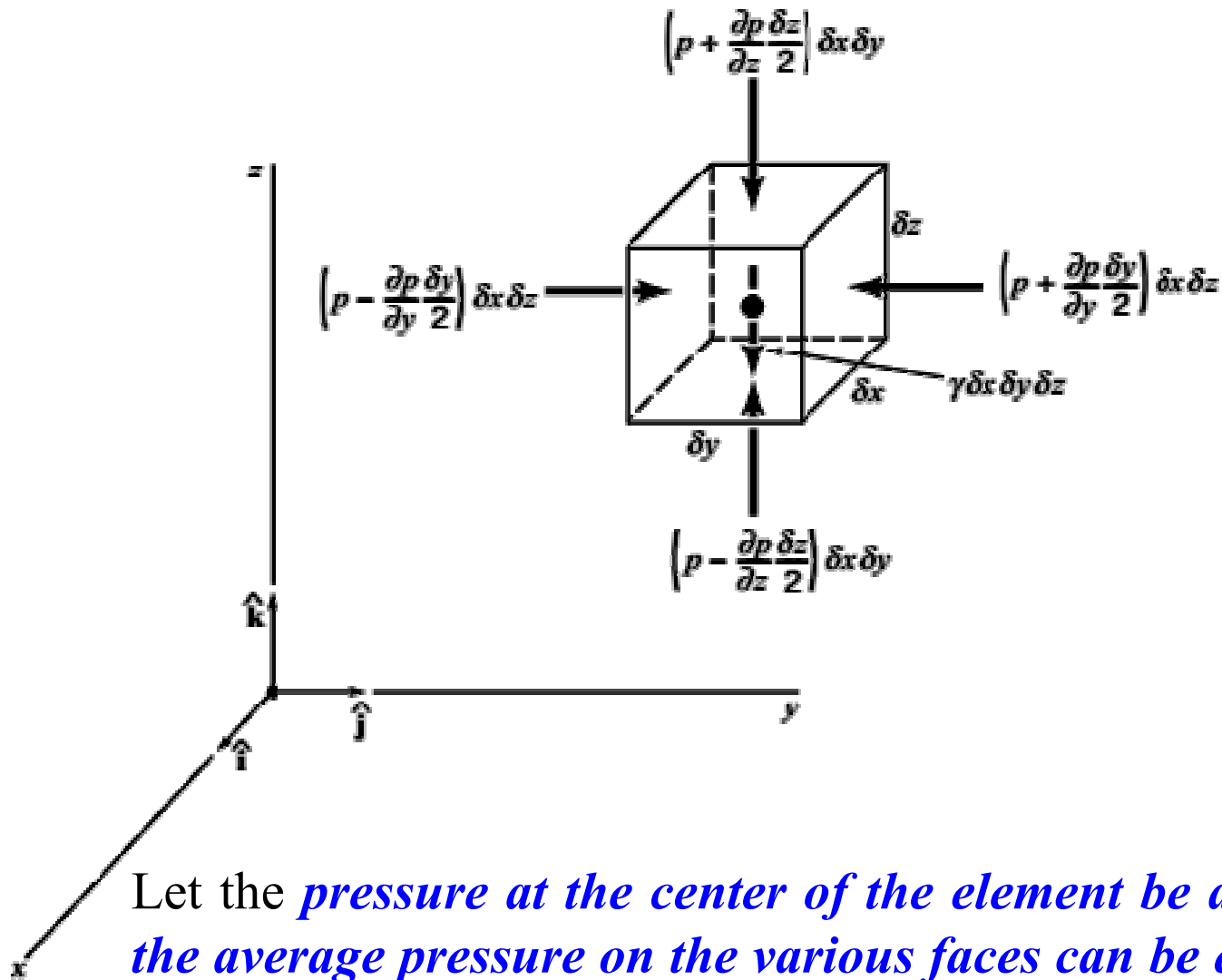
$$1 \text{ atm} = 14.696 \text{ psi}$$

## Basic Equation for Pressure Field

To predict how does the pressure variation of fluids at rest or fluids undergoing an acceleration while the relative position of fluid elements to one another remains the same i.e. in which there are no shearing stresses vary from point to point? –

Consider a small rectangular element of fluid removed from some arbitrary position within the mass of fluid of interest as shown in Fig where the z-axis is in the vertical direction. There are two types of forces acting on this element: surface forces due to the pressure, and a body force equal to the weight of the element (body forces, such as those due to magnetic fields is neglected.).





Let the *pressure at the center of the element be designated as  $p$* , then *the average pressure on the various faces can be expressed in terms of  $p$  and its derivatives*, as shown in Fig. *Using a Taylor series expansion* of the pressure with  $p(x, y, z)$  at the element center to approximate the pressures a short distance away and neglecting higher order terms that will vanish as we let  $\delta x$  and  $\delta y$  approach zero.

$$\boxed{dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz} \quad \text{----- (1)}$$

If we move from the center to a face a distance  $(dx/2)$  away, we see that the pressure is

$$\text{Rear face} \quad p\left(x - \frac{dx}{2}, y, z\right) = p(x, y, z) - \frac{\partial p}{\partial x} \frac{dx}{2}$$

$$\text{Front face} \quad p\left(x + \frac{dx}{2}, y, z\right) = p(x, y, z) + \frac{\partial p}{\partial x} \frac{dx}{2}$$

The **resultant surface force** in the x direction is

$$\delta F_x = \left(p - \frac{\partial p}{\partial x} \frac{dx}{2}\right) dy dz - \left(p + \frac{\partial p}{\partial x} \frac{dx}{2}\right) dy dz = -\frac{\partial p}{\partial x} dx dy dz \quad \text{----- (2a)}$$

Similarly, for the y and z directions the resultant surface forces are

$$\delta F_y = -\frac{\partial p}{\partial y} dx dy dz \quad \delta F_z = -\frac{\partial p}{\partial z} dx dy dz$$

----- (2b) ----- (2c)

## Taylor's Series method

Consider the one dimensional initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

where  $f$  is a function of two variables  $x$  and  $y$  and  $(x_0, y_0)$  is a known point on the solution curve.

If the existence of all higher order partial derivatives is assumed for  $y$  at  $x = x_0$ , then by Taylor series the value of  $y$  at any neighbouring point  $x+h$  can be written as

$$y(x_0+h) = y(x_0) + h y'(x_0) + \frac{h^2}{2} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \dots$$

where ' represents the derivative with respect to  $x$ .

$$y(x_0-h) = y(x_0) - h y'(x_0) + \frac{h^2}{2} y''(x_0) - \frac{h^3}{3!} y'''(x_0) + \dots$$

The **resultant surface force** acting on the element can be expressed in vector form as

$$\delta \mathbf{F}_s = \delta F_x \hat{\mathbf{i}} + \delta F_y \hat{\mathbf{j}} + \delta F_z \hat{\mathbf{k}} \quad \text{----- (2)}$$

The pressure variation from one point to another will be determined by applying **Newton's second law**; that is, *the sum of the forces acting on the fluid element is equal to the mass times the acceleration of the element (for constant mass system)*.

$$\Sigma \mathbf{F} = m\mathbf{a}$$

This results in the three component equations, assuming  $z$  to be vertical and using the mass as  $\rho \, dx \, dy \, dz$ ,

$$-\frac{\partial p}{\partial x} \, dx \, dy \, dz = \rho a_x \, dx \, dy \, dz \quad \text{----- (3a)}$$

$$-\frac{\partial p}{\partial y} \, dx \, dy \, dz = \rho a_y \, dx \, dy \, dz \quad \text{----- (3b)}$$

$$-\frac{\partial p}{\partial z} \, dx \, dy \, dz = \rho(a_z + g) \, dx \, dy \, dz \quad \text{----- (3c)}$$

where  $a_x$ ,  $a_y$ , and  $a_z$  are the components of the acceleration of the element. Division by the element's volume  $dx dy dz$  yields

$$\frac{\partial p}{\partial x} = -\rho a_x \quad \text{----- (4a)}$$

$$\frac{\partial p}{\partial y} = -\rho a_y \quad \text{----- (4b)}$$

$$\frac{\partial p}{\partial z} = -\rho(a_z + g) \quad \text{----- (4c)}$$

The pressure differential in any direction can now be determined from Eq. (1) as

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz$$

$$dp = -\rho a_x dx - \rho a_y dy - \rho(a_z + g) dz \quad \text{----- (4)}$$

where  $z$  is always vertical. Pressure differences between specified points can be found by integrating Eq. 4.

## Pressure Variation in a Fluid at Rest

*A fluid at rest does not undergo any acceleration.* Therefore, set  $a_x = a_y = a_z = 0$  and Eq. (4) reduces to

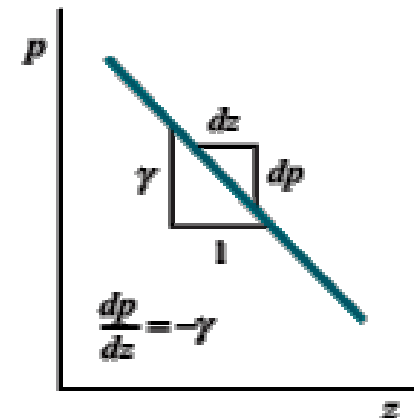
$$\boxed{\frac{\partial p}{\partial x} = 0 \quad \frac{\partial p}{\partial y} = 0 \quad \frac{\partial p}{\partial z} = -\rho g} \quad \text{----- (5)}$$

Pressure in a fluid at rest is independent of the shape or cross section of the container. Thus pressure in a fluid at rest does not change in the horizontal direction. This can be shown easily by considering a thin horizontal layer of fluid and doing a force balance in any horizontal direction. Pressure in a fluid increases with depth (vertical direction in a gravity field) because more fluid rests on deeper layers, and the effect of this “extra weight” on a deeper layer is balanced by an increase in pressure.

$$\boxed{\frac{dp}{dz} = -\rho g \equiv -\gamma} \quad \gamma \text{ is the specific weight of the fluid.}$$

This equation is the basic hydrostatic (pressure height) relation of fluid statics.

The negative sign is due to our taking the positive  $z$  direction to be upward so that  $dP$  is negative when  $dz$  is positive since pressure decreases in an upward direction.



**Restrictions:** (1) Static fluid. (2) Gravity is the only body force.  
(3) The  $z$  axis is vertical and upward.

✚ For small to moderate distances, the variation of pressure with height is negligible for gases because of their low density. The pressure in a tank containing a gas, for example, can be considered to be uniform since the weight of the gas is too small to make a significant difference. Also, the pressure in a room filled with air can be approximated as a constant.

## Pressures in Incompressible ( $\rho = \text{constant}$ ) Fluid at Rest

Since the specific weight is equal to the product of fluid density and acceleration of gravity ( $\gamma = \rho g$ ), *changes in  $\gamma$  are caused either by a change in  $\rho$  or  $g$* . For most engineering applications *the variation in  $g$  is negligible*, so our main concern is with the possible variation in the fluid density. In general, *a fluid with constant density is called an incompressible fluid*. *For liquids the variation in density is usually negligible*, even over large vertical distances. For this instance, Eq. (5) can be directly integrated

$$\int_{p_1}^{p_2} dp = -\gamma \int_{z_1}^{z_2} dz$$

to yield

$$p_2 - p_1 = -\gamma(z_2 - z_1)$$

Or,

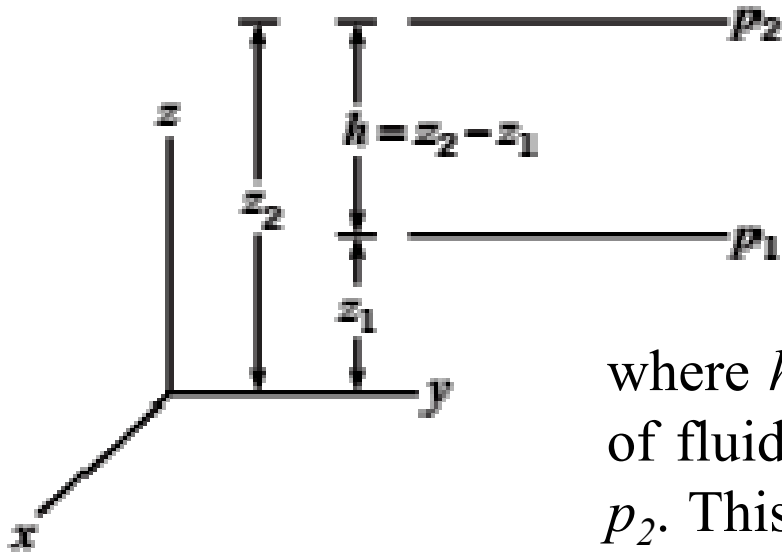
$$p_1 - p_2 = \gamma(z_2 - z_1)$$

$$\Delta P = p_2 - p_1 = -\int_1^2 \rho g dz$$

where  $p_1$  and  $p_2$  are pressures at the vertical elevations  $z_1$  and  $z_2$  as is illustrated in following Fig.

$$p_2 - p_1 = -\gamma(z_2 - z_1)$$

Free surface  
(pressure =  $p_0$ )



Equation (5) can be written in the compact form

$$p_1 - p_2 = \gamma h$$

$$p_1 = \gamma h + p_2 \quad \text{----- (6)}$$

where  $h$  is the distance,  $z_2 - z_1$  which is the depth of fluid measured downward from the location of  $p_2$ . This type of pressure distribution is commonly called *hydrostatic distribution*.

Eq. (6) shows that in *an incompressible fluid at rest the pressure varies linearly with depth*. **The pressure must increase with depth to “hold up” the fluid above it.** The pressure difference between two points can be specified by the distance  $h$  since

$$h = \frac{p_1 - p_2}{\gamma}$$

$h$  is called the *pressure head*

$$\Delta p = -\gamma \Delta z \quad \text{or} \quad p + \gamma z = \text{constant} \quad \text{or} \quad \frac{p}{\gamma} + z = \text{constant}$$

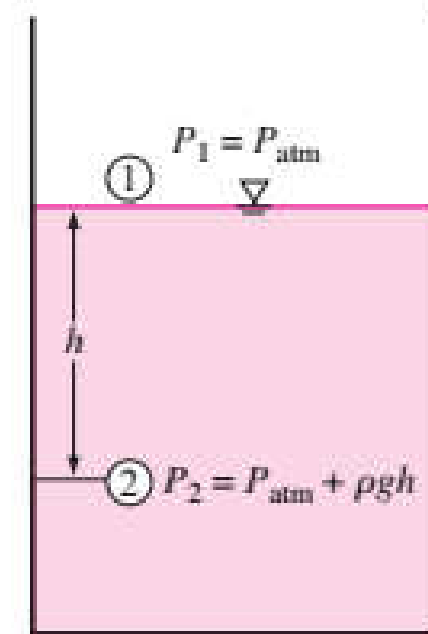
The quantity  $(p/\gamma + z)$  is often referred to as the piezometric head.

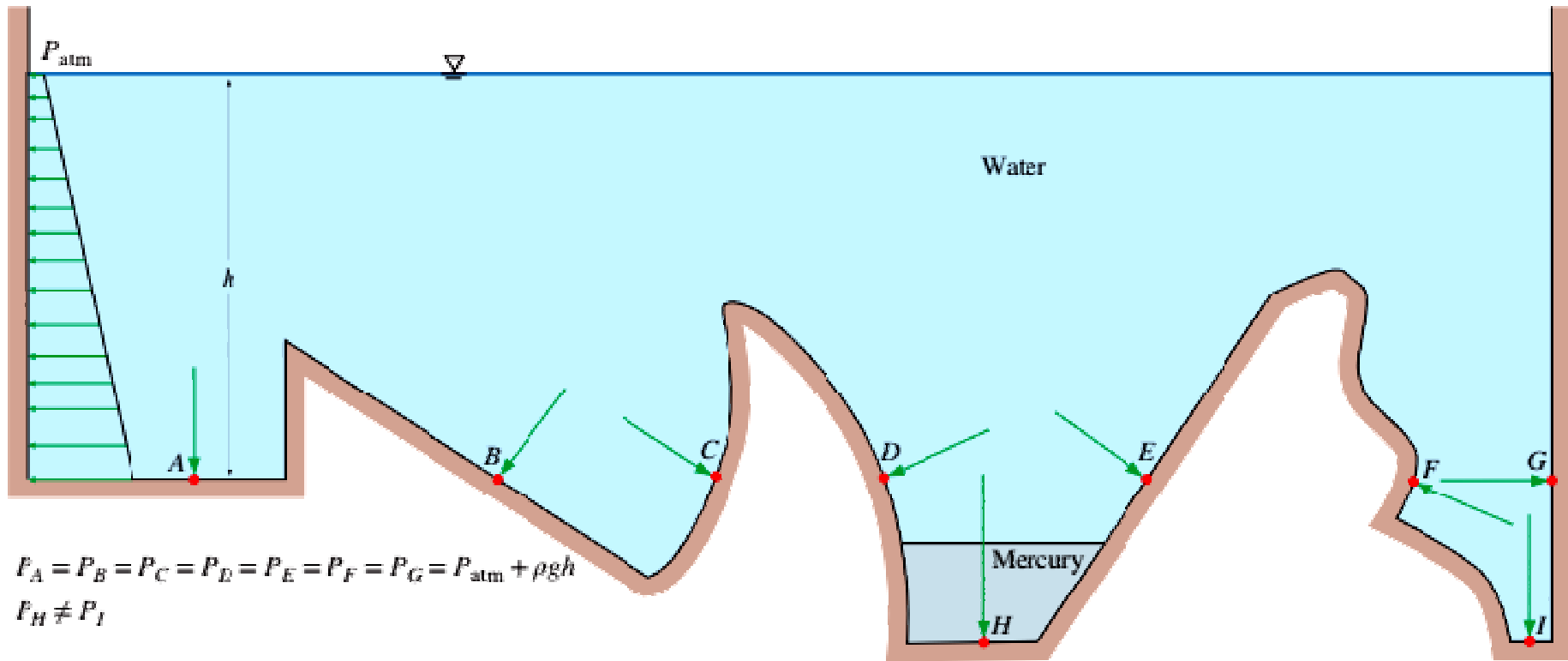
✚ If we take point 1 to be at the **free surface (A surface separating a gas from a liquid)** of a liquid open to the atmosphere as shown in Fig., where the pressure is the atmospheric pressure  $P_{\text{atm}}$ , then the pressure at a depth  $h$  from the free surface becomes

$$P = P_{\text{atm}} + \rho gh \quad \text{or} \quad P_{\text{gage}} = \rho gh$$

So the pressure at any point in a static fluid is the summation of atmospheric pressure at the free surface and the pressure due to distance of that point below the free surface. If the atmospheric pressure is considered as zero of pressure scale, then  $p = \gamma h$

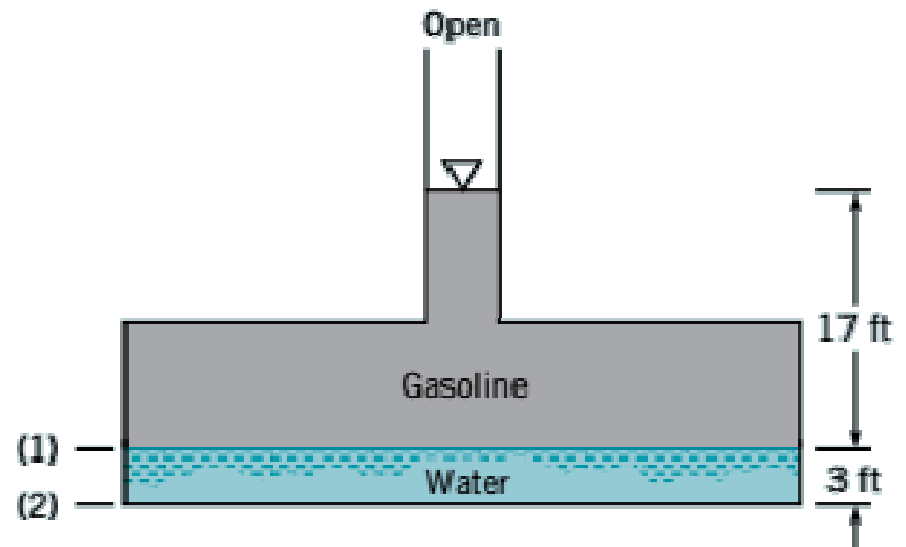
This pressure is the **positive or gauge pressure** which is above atmospheric pressure.





**Problem:**

Because of a leak in a buried gasoline storage tank, water has seeped in to the depth shown in Fig. The specific gravity of the gasoline is  $SG = 0.68$ .



Determine the pressure at the gasoline–water interface and at the bottom of the tank. Express the pressure in units of lb/ft<sup>2</sup>, lb/in.<sup>2</sup>, and as a pressure head in feet of water.

**Solution:**

The pressure variation in liquids at rest can be found from the equation:

$$p = \gamma h + p_0$$

With  $p_0$  corresponding to the pressure at the free surface of the gasoline, then the pressure at the interface is

$$\begin{aligned} p_1 &= SG\gamma_{H_2O}h + p_0 \\ &= (0.68)(62.4 \text{ lb/ft}^3)(17 \text{ ft}) + p_0 \\ &= 721 + p_0 \text{ (lb/ft}^2\text{)} \end{aligned}$$

If we measure the pressure relative to atmospheric pressure (gage pressure), it follows that  $p_0 = 0$  and therefore

$$p_1 = 721 \text{ lb/ft}^2 \quad \text{(Ans)}$$

$$p_1 = \frac{721 \text{ lb/ft}^2}{144 \text{ in.}^2/\text{ft}^2} = 5.01 \text{ lb/in.}^2 \quad \text{(Ans)}$$

$$\frac{p_1}{\gamma_{\text{H}_2\text{O}}} = \frac{721 \text{ lb/ft}^2}{62.4 \text{ lb/ft}^3} = 11.6 \text{ ft} \quad \text{(Ans)}$$

It is noted that a rectangular column of water 11.6 ft tall and 1 ft<sup>2</sup> in cross-section weighs 721 lb. A similar column with a 1 in<sup>2</sup> cross-section weighs 5.01 lb.

Now apply the same relationship to determine the pressure at the tank bottom; that is,

$$\begin{aligned} p_2 &= \gamma_{\text{H}_2\text{O}} h_{\text{H}_2\text{O}} + p_1 \\ &= (62.4 \text{ lb/ft}^3)(3 \text{ ft}) + 721 \text{ lb/ft}^2 \quad \text{(Ans)} \\ &= 908 \text{ lb/ft}^2 \end{aligned}$$

$$p_2 = \frac{908 \text{ lb/ft}^2}{144 \text{ in.}^2/\text{ft}^2} = 6.31 \text{ lb/in.}^2 \quad \text{(Ans)}$$

$$\frac{p_2}{\gamma_{\text{H}_2\text{O}}} = \frac{908 \text{ lb/ft}^2}{62.4 \text{ lb/ft}^3} = 14.6 \text{ ft} \quad \text{(Ans)}$$

Observe that if we wish to express these pressures in terms of **absolute pressure**, we would have to add the local atmospheric pressure (in appropriate units) to the previous results.

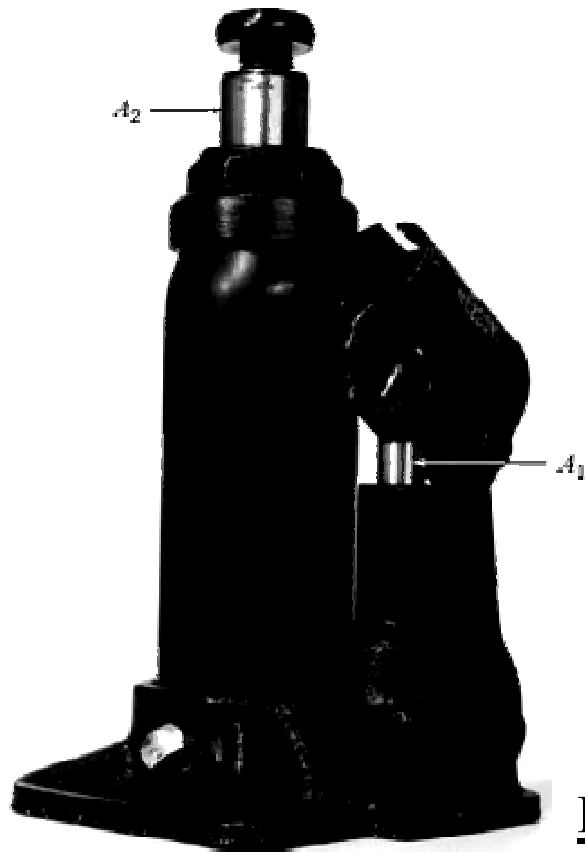


Fig. a

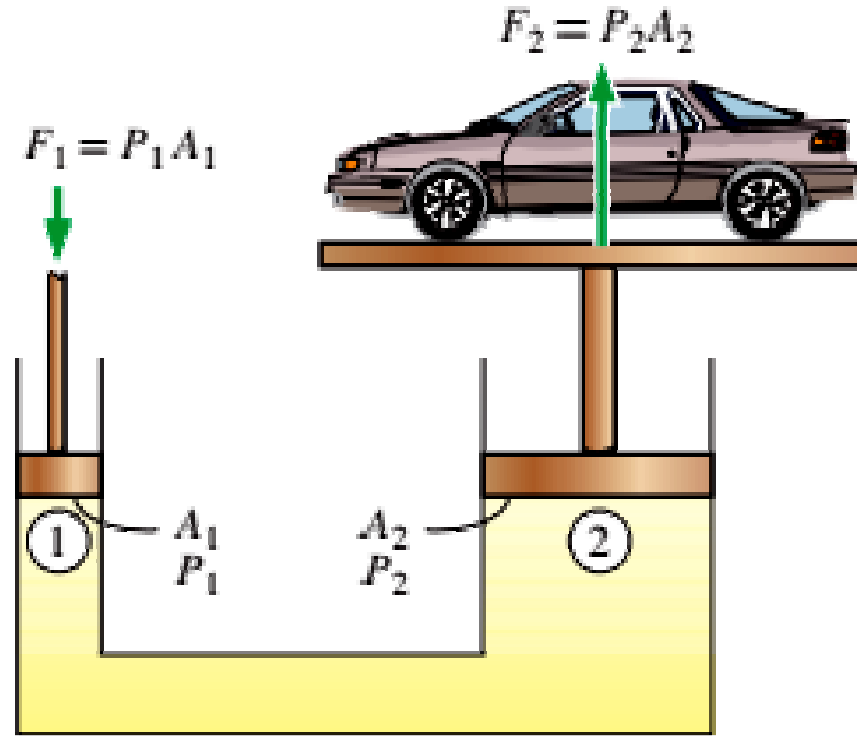


Fig. b

A consequence of the pressure in a fluid remaining constant in the horizontal direction *is that the pressure applied to a confined fluid increases the pressure throughout by the same amount (Pascal's law). Pascal knew that the force applied by a fluid is proportional to the surface area.* He realized that two hydraulic cylinders of different areas could be connected, and the larger could be used to exert a proportionally greater force than that applied to the smaller.

The transmission of pressure throughout a stationary fluid is the principle upon which many hydraulic devices are based.

The required equality of pressures at equal elevations throughout a system is important for the operation of hydraulic jacks (Fig. *a*), lifts, and presses, as well as hydraulic controls on aircraft and other types of heavy machinery. This enables us to lift a car easily by one arm, as shown in Fig b.

Noting that  $P_1 = P_2$  since both pistons are at the same level (the effect of small height differences is negligible, especially at high pressures), the ratio of output force to input force is determined to be

$$P_1 = P_2 \quad \rightarrow \quad \frac{F_1}{A_1} = \frac{F_2}{A_2} \quad \rightarrow \quad \frac{F_2}{F_1} = \frac{A_2}{A_1}$$

The area ratio  $A_2/A_1$  is called the *ideal mechanical advantage* of the hydraulic lift. Using a hydraulic car jack with a piston area ratio of  $A_2/A_1 = 10$ , for example, a person can lift a 1000-kg car by applying a force of just 100 kgf (= 908 N).

### **Problem:**

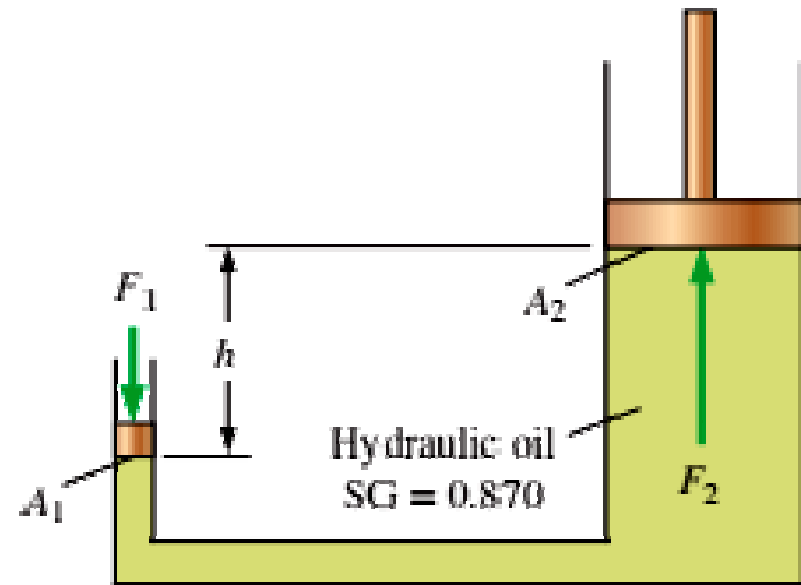
Consider a hydraulic jack being used in a car repair shop, as in Fig. The pistons have an area of  $A_1 = 0.8 \text{ cm}^2$  and  $A_2 = 0.04 \text{ m}^2$ . Hydraulic oil with a specific gravity of 0.870 is pumped in as the small piston on the left side is pushed up and down, slowly raising the larger piston on the right side. A car that weighs 13,000 N is to be jacked up.

- (a) At the beginning, when both pistons are at the same elevation ( $h = 0$ ), calculate the force  $F_1$  in newtons required to hold the weight of the car.
- (b) Repeat the calculation after the car has been lifted two meters ( $h = 2 \text{ m}$ ). Compare and discuss.

### **Solution:**

#### **Assumptions:**

1. The oil is incompressible.
2. The system is at rest during the analysis (hydrostatics).
3. Air density is negligible compared to oil density.



**(a) When  $h = 0$** , the pressure at the bottom of each piston must be the same. Thus,

$$P_1 = \frac{F_1}{A_1} = P_2 = \frac{F_2}{A_2} \rightarrow F_1 = F_2 \frac{A_1}{A_2} = (13,000 \text{ N}) \frac{0.8 \text{ cm}^2}{0.0400 \text{ m}^2} \left( \frac{1 \text{ m}}{100 \text{ cm}} \right)^2 = 26.0 \text{ N}$$

**(b) When  $h \neq 0$** , the hydrostatic pressure due to the elevation difference must be taken into account, namely,

$$P_1 = \frac{F_1}{A_1} = P_2 + \rho gh = \frac{F_2}{A_2} + \rho gh$$

$$F_1 = F_2 \frac{A_1}{A_2} + \rho gh A_1$$

$$= (13,000 \text{ N}) \frac{0.00008 \text{ m}^2}{0.04 \text{ m}^2}$$

$$+ (870 \text{ kg/m}^3)(9.807 \text{ m/s}^2)(2.00 \text{ m})(0.00008 \text{ m}^2) \left( \frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2} \right) = 27.4 \text{ N}$$

Comparing the two results, it takes more force to keep the car elevated than it does to hold it at  $h = 0$ . This makes sense physically because the elevation difference generates a higher pressure (and thus a higher required force) at the lower piston due to hydrostatics.

### **Discussion**

When  $h = 0$ , the specific gravity (or density) of the hydraulic fluid does not enter the calculation—the problem simplifies to setting the two pressures equal. However, when  $h \neq 0$ , there is a hydrostatic head and therefore the density of the fluid enters the calculation. The air pressure on the right side of the jack is actually slightly lower than that on the left side, but we have neglected this effect.

## Pressures in Compressible ( $\rho \neq \text{constant}$ ) Fluid

Gases such as air, oxygen, and nitrogen are being **compressible fluids** since the density of the gas can change significantly with changes in pressure and temperature. Since the specific weights of gases are comparatively small, it follows from

$$\boxed{\frac{dp}{dz} = -\gamma} \quad \text{----- (5)}$$

that the pressure gradient in the vertical direction is correspondingly small, and even over distances of several hundred feet the pressure will remain essentially constant *for a gas*. We can neglect the effect of elevation changes on the pressure in gases in tanks, pipes, and so forth in which the distances involved are small.

For those situations in which the variations in heights are large, on the order of thousands of feet, the variation in the specific weight is not negligible.

The equation of state for an ideal (or perfect) gas is

$$p = \frac{P}{RT}$$

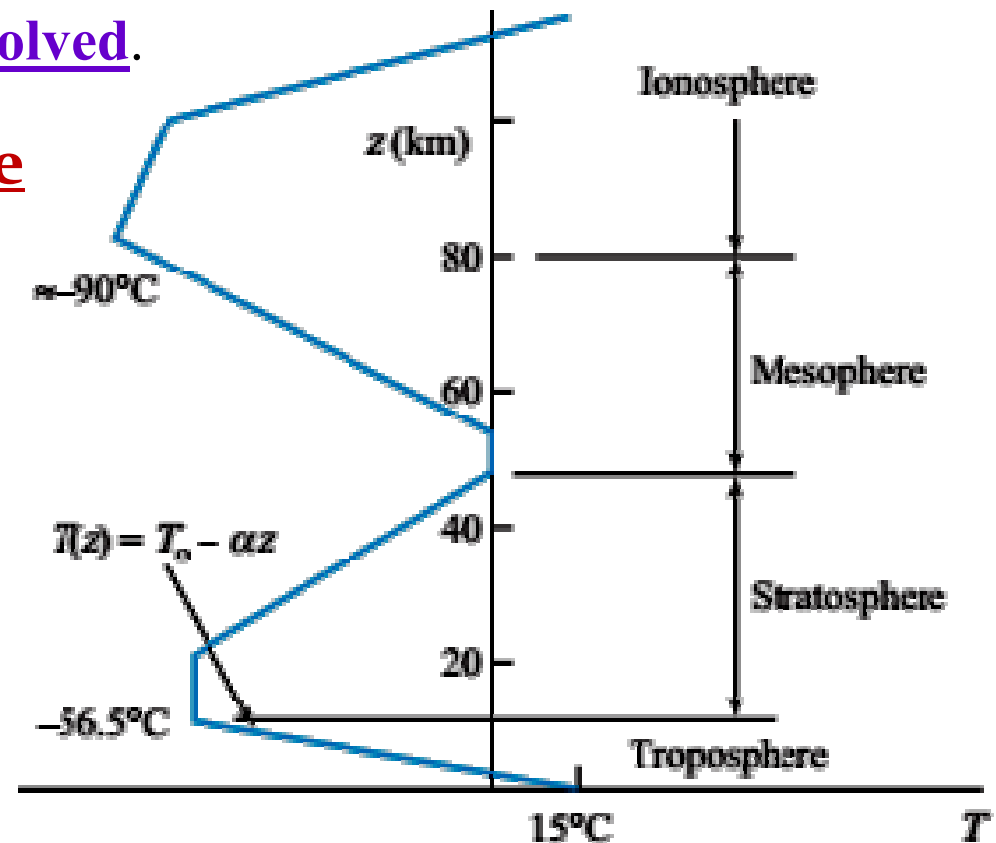
where  $p$  is the absolute pressure,  $R$  is the gas constant, and  $T$  is the absolute temperature. *From hydrostatic equation,*

$$dp = -\rho g dz = -\frac{P g}{RT} dz \quad \text{----- (7)}$$

Variation of pressure can be evaluated depending upon the process involved.

## Pressures in the Atmosphere

For the atmosphere where the *density depends on height* [i.e.,  $\rho = \rho(z)$ ], we must integrate Eq. (7) along a vertical path. *The atmosphere is divided into four layers:* the *troposphere* (nearest Earth), the *stratosphere*, the *mesosphere*, and the *ionosphere*.



## (i) Variation Of Pressure and Density With Altitude For a Constant Temperature Gradient

Because conditions change with time and latitude in the atmosphere with the layers being thicker at the equator and thinner at the poles, we base calculations on the **standard atmosphere, which is at 40° latitude**. In the standard atmosphere the temperature in the troposphere varies linearly with elevation,  $T(z) = T_0 - mz$ , where *the lapse rate*  $m = 0.0065$  K/m (0.00357°R/ft) and  $T_0$  is 288 K (518°R).

$$dp = -\rho g dz = -\frac{p g}{RT} dz = -\frac{p g}{R(T_0 - mz)} dz$$

Separating variables and integrating from  $z = 0$  where  $p = p_0$  to elevation  $z$  where the pressure is  $p$  gives

$$\int_{p_0}^p \frac{dp}{p} = - \int_0^z \frac{g dz}{R(T_0 - mz)}$$

Then

$$\ln \frac{p}{p_0} = \frac{g}{mR} \ln \left( \frac{T_0 - mz}{T_0} \right) = \frac{g}{mR} \ln \left( 1 - \frac{mz}{T_0} \right)$$

$$p = p_0 \left( 1 - \frac{mz}{T_0} \right)^{g/mR} = p_0 \left( \frac{T}{T_0} \right)^{g/mR}$$

## (ii) For Isothermal Process Temperature is Constant

i.e.  $T = T_0 = \text{constant}$

In the lower part of the stratosphere, the temperature is constant. This equation can be integrated from point 1 to point 2, yielding

$$dp = -\rho g dz = -\frac{p g}{RT} dz \quad \int_{p_1}^{p_2} \frac{dp}{p} = -\frac{g}{RT} \int_{z_1}^{z_2} dz \quad \text{Or, } \ln \frac{p_2}{p_1} = -\frac{g}{RT} (z_2 - z_1)$$

Rearranging, we get

$$z_2 - z_1 = -\frac{RT}{g} \ln \frac{p_2}{p_1}$$

$$p_2/p_1 = \exp[-(g/RT)(z_2 - z_1)]$$

## (iii) Variation of Pressure with Altitude in a Gas Under Adiabatic Conditions

If conditions are adiabatic, the relationship between pressure and density is given by  $p/\rho^\gamma = \text{constant} = p_1/\rho_1^\gamma$ , so that

$$\rho = \rho_1 (p/p_1)^{1/\gamma}$$

Substituting in equation (7),

$$dp = -\rho g dz = -\frac{\rho g}{RT} dz \quad \text{----- (7)}$$

$$\frac{dp}{dz} = -\frac{\rho_1 g}{p_1^{1/\gamma}} p^{1/\gamma}, \quad \rho = \rho_1 (p/p_1)^{1/\gamma}$$

$$dz = -\left(\frac{p_1^{1/\gamma}}{\rho_1 g}\right) p^{-1/\gamma} dp$$

Integrating from  
 $p = p_1$  when  $z = z_1$ , to  
 $p = p_2$  when  $z = z_2$

$$\begin{aligned} z_2 - z_1 &= -\frac{p_1^{1/\gamma}}{\rho_1 g} \left[ \frac{p^{(\gamma-1)/\gamma}}{(\gamma-1)/\gamma} \right]_{p_1}^{p_2} \\ &= -\left(\frac{\gamma}{\gamma-1}\right) \frac{p_1^{1/\gamma}}{\rho_1 g} (p_2^{(\gamma-1)/\gamma} - p_1^{(\gamma-1)/\gamma}) \\ &= -\left(\frac{\gamma}{\gamma-1}\right) \frac{p_1}{\rho_1 g} \left[ \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} - 1 \right], \end{aligned}$$

or, since  $p_1/\rho_1 = RT$ , for any gas,

$$z_2 - z_1 = -\left(\frac{\gamma}{\gamma-1}\right) \frac{RT_1}{g} \left[ \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} - 1 \right]$$

or,

$$\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} = -\frac{g(z_2 - z_1)(\gamma-1)}{RT_1} + 1$$

$$\frac{p_2}{p_1} = \left[ 1 - \frac{g(z_2 - z_1)(\gamma - 1)}{RT_1} \right]^{\gamma/(\gamma - 1)}$$

This can be extended to any *isentropic process* for which  $p/\rho^n = \text{constant}$ , to give

$$\frac{p_2}{p_1} = \left[ 1 - \frac{g(z_2 - z_1)(n - 1)}{RT_1} \right]^{n/(n - 1)}$$

- The rate of change of temperature with altitude – the **temperature lapse rate,  $m$**  – can also be found for adiabatic conditions. From the characteristic equation,  $\rho = p/RT$  and, since, from equation

$$dz = -dp/\rho g,$$

substituting for  $\rho$ ,  $dz = -(RT/gp) dp$

$$\rho = \frac{p}{RT}$$

For adiabatic conditions,  $p/\rho^\gamma = p_1/\rho_1^\gamma$

or, since  $p/\rho = RT$ , For reversible adiabatic processes, it is also true that

$$p^{1-\gamma} T^\gamma = \text{constant}$$

$$p = p_1 (T_1/T)^{\gamma/(1-\gamma)}$$

$$\gamma = \frac{C_p}{C_v}$$

and, differentiating,  $dp = -[\gamma/(1-\gamma)] p_1 T^{\gamma/(1-\gamma)} T^{-\gamma/(1-\gamma)} dT$

In thermodynamics, an **isentropic process** is an idealized thermodynamic process that is adiabatic and in which the work transfers of the system are frictionless; there is no transfer of heat or matter and the process is reversible. Such an idealized process is useful in engineering as a model of and basis of comparison for real processes.

Substituting these values of  $p$  and  $dp$  in equation  $dz$

$$\begin{aligned} dz &= -\frac{RT}{g} \frac{\{-[\gamma/(1-\gamma)] p_1 T^{\gamma/(1-\gamma)} T^{-\gamma/(1-\gamma)} dT\}}{p_1 T^{\gamma/(1-\gamma)} T^{-\gamma/(1-\gamma)}} \\ &= [\gamma/(1-\gamma)] (R/g) dT \end{aligned}$$

Temperature gradient,  $m$   
(Lapse rate)

$$\frac{dT}{dz} = -[(\gamma - 1)/\gamma](g/R)$$

**Problem:**

Calculate the pressure, temperature and density of the atmosphere at an altitude of 1200 m if at zero altitude the temperature is 15°C and the pressure 101 kNm<sup>-2</sup>. Assume that conditions are adiabatic ( $\gamma = 1.4$ ) and  $R = 287 \text{ Jkg}^{-1}\text{K}^{-1}$ .

**Solution:**

For adiabatic conditions

$$\frac{p_2}{p_1} = \left[ 1 - \frac{g(z_2 - z_1)(\gamma - 1)}{RT_1} \right]^{\gamma/(\gamma - 1)}$$

Putting  $p_1 = 101 \times 10^3 \text{ Nm}^{-2}$ ,  $z_1 = 0$ ,  $z_2 = 1200 \text{ m}$ ,  $T_1 = 15^\circ\text{C} = 288 \text{ K}$ ,  
 $\gamma = 1.4$ ,  $R = 287 \text{ Jkg}^{-1}\text{K}^{-1}$

$$\begin{aligned} p_2 &= 101 \times 10^3 \left[ 1 - \frac{9.81 \times 1200}{287 \times 288} \left( \frac{0.4}{1.4} \right) \right]^{1.4/0.4} \text{ Nm}^{-2} \\ &= 87.33 \times 10^3 \text{ Nm}^{-2}. \end{aligned}$$

Temperature gradient,

$$\begin{aligned}\frac{dT}{dz} &= -[(\gamma-1)/\gamma](g/R) = -(0.4/1.4) \times (9.81/287) \\ &= -9.76 \times 10^{-3} \text{ K m}^{-1}, \\ T_2 &= T_1 + \frac{dT}{dz} (z_2 - z_1) \\ &= 288 - (9.76 \times 10^{-3} \times 1200) = 276.3 \text{ K} = 3.3^\circ\text{C}.\end{aligned}$$

From the equation of state, Density at 1200 m,

$$\begin{aligned}\rho_2 &= p_2/RT_2 \\ &= (87.33 \times 10^3)/(287 \times 276.3) = 1.101 \text{ kg m}^{-3}.\end{aligned}$$

### **Problem:**

The maximum power output capability of a gasoline or diesel engine decreases with altitude because the air density and hence the mass flow rate of air decrease. A truck leaves Denver (elevation 5280 ft) on a day when the local temperature and barometric pressure are 80°F and 24.8 in. of mercury, respectively. It travels through Vail Pass (elevation 10,600 ft), where the temperature is 62°F. Determine the local barometric pressure at Vail Pass and the percent change in density.

***Given:*** Truck travels from Denver to Vail Pass.

Denver:  $z = 5280$  ft

$p = 24.8$  in. Hg

$T = 80^\circ\text{F}$

Vail Pass:  $z = 10,600$  ft

$T = 62^\circ\text{F}$

### **Solution:**

**Governing equations:**  $\frac{dp}{dz} = -\rho g$      $p = \rho RT$

**Assumptions:** (1) Static fluid. (2) Air behaves as an ideal gas.

Consider four assumptions for property variations with altitude.

*(a) If we assume temperature varies linearly with altitude*

$$\frac{P}{P_0} = \left( \frac{T}{T_0} \right)^{g/mR}$$

$$T(z) = T_0 - mz = T_0 - m(z-z_0)$$

Evaluating the constant  $m$  gives

$$m = \frac{T_0 - T}{z - z_0} = \frac{(80 - 62)^\circ\text{F}}{(10.6 - 5.28)10^3 \text{ ft}} = 3.38 \times 10^{-3} \text{ }^\circ\text{F/ft}$$

and

$$\frac{g}{mR} = 32.2 \frac{\text{ft}}{\text{s}^2} \times \frac{\text{ft}}{3.38 \times 10^{-3} \text{ }^\circ\text{F}} \times \frac{\text{lbm} \cdot \text{ }^\circ\text{R}}{53.3 \text{ ft} \cdot \text{ lbf}} \times \frac{\text{slug}}{32.2 \text{ lbm}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} = 5.55$$

Thus

$$\frac{P}{P_0} = \left( \frac{T}{T_0} \right)^{g/mR} = \left( \frac{460 + 62}{460 + 80} \right)^{5.55} = (0.967)^{5.55} = 0.830$$

And  $p = 0.830 p_0 = (0.830)24.8 \text{ in. Hg} = 20.6 \text{ in. Hg}$

The percent change in density is given by

$$\frac{\rho - \rho_0}{\rho_0} = \frac{\rho}{\rho_0} - 1 = \frac{P}{P_0} \frac{T_0}{T} - 1 = \frac{0.830}{0.967} - 1 = -0.142 \text{ or } -14.2\%$$

**(b) For  $\rho$  assumed constant ( $= \rho_0$ )**

$$p = p_0 - \rho_0 g(z - z_0) = p_0 - \frac{\rho_0 g(z - z_0)}{RT_0} = p_0 \left[ 1 - \frac{g(z - z_0)}{RT_0} \right]$$

$$p = 20.2 \text{ in. Hg} \quad \text{and} \quad \frac{\Delta p}{p_0} = 0$$

**(c) If we assume the temperature is constant, then**

$$dp = -\rho g dz = -\frac{p}{RT} g dz$$
$$\int_{p_0}^p \frac{dp}{p} = - \int_{z_0}^z \frac{g}{RT} dz$$
$$p = p_0 \exp \left[ \frac{-g(z - z_0)}{RT} \right]$$

For  $T = \text{constant} = T_0$ ,  $p = 20.6 \text{ in. Hg}$  and  $\frac{\Delta p}{p_0} = -16.9\%$

**(d) For an adiabatic atmosphere  $p/\rho^k = \text{constant}$ ,**

$$p = p_0 \left( \frac{T}{T_0} \right)^{k/k-1} = 22.0 \text{ in. Hg} \quad \text{and} \quad \frac{\Delta p}{p_0} = -8.2\%$$

Over the modest change in elevation the predicted pressure is not strongly dependent on the assumed property variation; values calculated under four different assumptions vary by a maximum of approximately 9 percent. There is considerably greater variation in the predicted percent change in density. The assumption of a linear temperature variation with altitude is the most reasonable assumption

<b>R for dry air</b>	<b>Units</b>
287.058	J kg <sup>-1</sup> K <sup>-1</sup>
53.3533	ft <u>lbf lb</u> <sup>-1</sup> °R <sup>-1</sup>
1,716.49	ft <u>lbf slug</u> <sup>-1</sup> °R <sup>-1</sup>
Based on a mean molar mass for dry air of 28.9645 g/mol.	

The slug is a derived unit of mass in the weight-based system of measures. A slug is defined as the mass that is accelerated by 1ft/s<sup>2</sup> when a force of one pound (lb<sub>F</sub>) is exerted on it.

$$1 \text{ slug} = 1 \frac{\text{lb}_F \cdot \text{s}^2}{\text{ft}}$$

$$\text{or, } 1 \text{ lb}_F = 1 \frac{\text{slug} \cdot \text{ft}}{\text{s}^2}$$

One slug has a mass of 32.174049 lb<sub>m</sub> or 14.593903 kg based on standard gravity. At the surface of the Earth, an object with a mass of 1 slug exerts a force downward (the definition of weight) of approximately 32.2 lb<sub>F</sub> or 143 N.

## Pressure Measurement By Manometer

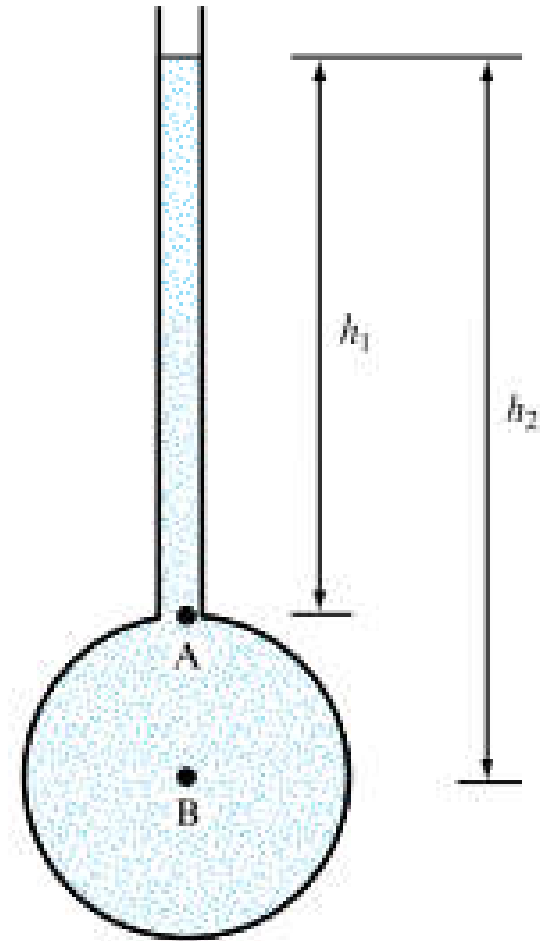
*A device which measures the fluid pressure by the height of a liquid column is called a manometer.* The relationship between pressure and head is utilized for pressure measurement in the manometer or liquid gauge.

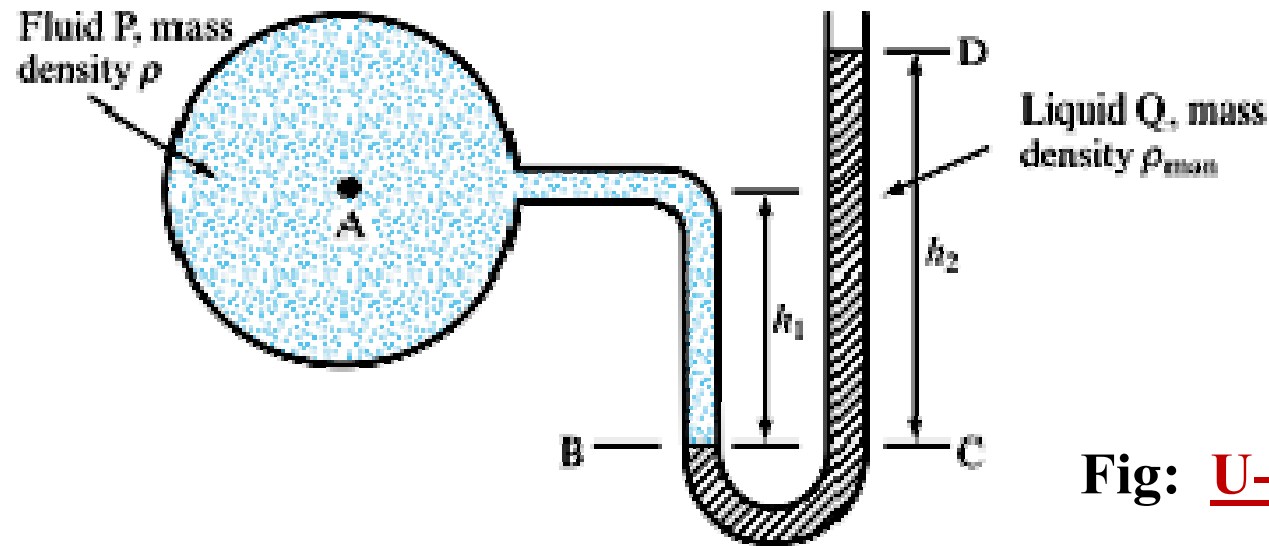
The simplest form is the pressure tube or piezometer shown in Fig. consisting of a single vertical tube, open at the top, inserted into a pipe or vessel containing liquid under pressure which rises in the tube to a height depending on the pressure. If the top of the tube is open to the atmosphere, the pressure measured is 'gauge' pressure:

Pressure at A = Pressure due to column of liquid of height  $h_1$ ,

$$p_A = \rho g h_1$$

Similarly, Pressure at B =  $p_B = \rho g h_2$ .





**Fig: U-tube manometer**

The U-tube gauge, shown in Fig. can be used to measure the pressure of either liquids or gases. The bottom of the U-tube is filled with a manometric liquid Q which is of greater density  $\rho_{man}$  and is immiscible with the fluid P, liquid or gas, of density  $\rho$ , whose pressure is to be measured. If B is the level of the interface in the left hand limb and C is a point at the same level in the right-hand limb,

$$\text{Pressure } p_B \text{ at B} = \text{Pressure } p_C \text{ at C}$$

For the left-hand limb,

$$\begin{aligned} p_B &= \text{Pressure } p_A \text{ at } A + \text{Pressure due to depth } h_1 \text{ of fluid } P \\ &= p_A + \rho g h_1 \end{aligned}$$

For the right-hand limb,

$$p_C = \text{Pressure } p_D \text{ at } D + \text{Pressure due to depth } h_2 \text{ of liquid } Q$$

But  $p_D = \text{Atmospheric pressure} = \text{Zero gauge pressure},$

and so  $p_C = 0 + \rho_{man}gh_2$

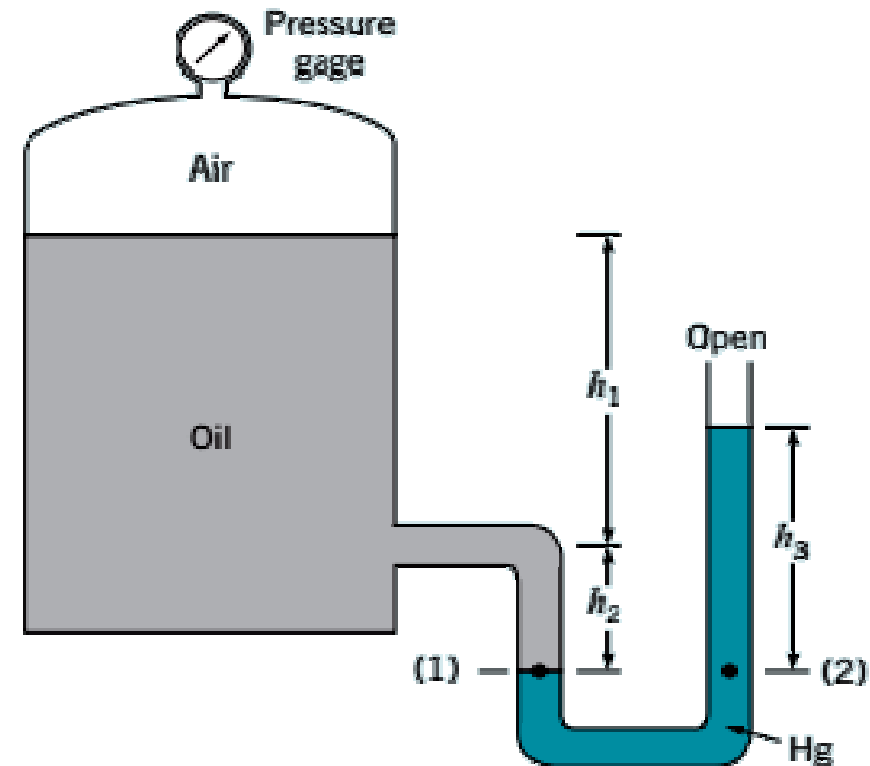
Since  $p_B = p_C,$

$$p_A + \rho gh_1 = \rho_{man}gh_2$$

$$p_A = \rho_{man}gh_2 - \rho gh_1$$

### **Problem:**

A closed tank contains compressed air and oil ( $SG_{oil} = 0.9$ ) as is shown in Fig. A U-tube manometer using mercury ( $SG_{Hg} = 13.6$ ) is connected to the tank as shown. The column heights are  $h_1 = 36$  in.,  $h_2 = 6$  in., and  $h_3 = 9$  in. Determine the pressure reading (in psi) of the gage.



### Solution:

Pressure at the air–oil interface in the tank and at the open end where the pressure is zero. **The pressure at level (1) is**

$$P_1 = P_{\text{air}} + \gamma_{\text{oil}}(h_1 + h_2)$$

**This pressure is equal to the pressure at level (2)**, since these two points are at the same elevation in a homogeneous fluid at rest. As we move from level (2) to the open end, the pressure must decrease by and at the open end the pressure is zero. Thus, the manometer equation can be expressed as

$$P_{\text{air}} + \gamma_{\text{oil}}(h_1 + h_2) - \gamma_{\text{Hg}}h_3 = 0$$

Or, 
$$P_{\text{air}} + (SG_{\text{oil}})(\gamma_{\text{H}_2\text{O}})(h_1 + h_2) - (SG_{\text{Hg}})(\gamma_{\text{H}_2\text{O}})h_3 = 0$$

$$P_{\text{air}} = -(0.9)(62.4 \text{ lb/ft}^3) \left( \frac{36 + 6}{12} \text{ ft} \right) \\ + (13.6)(62.4 \text{ lb/ft}^3) \left( \frac{9}{12} \text{ ft} \right)$$

$$P_{\text{air}} = 440 \text{ lb/ft}^2$$

$$P_{\text{gage}} = \frac{440 \text{ lb/ft}^2}{144 \text{ in.}^2/\text{ft}^2} = 3.06 \text{ psi}$$

Manometers are often used to measure the difference in pressure between two points. The U-tube manometer is also widely used to measure the difference in pressure between two containers or two points in a given system.

Consider a manometer connected between containers A and B as is shown in Fig. The difference in pressure between A and B can be found by again starting at one end of the system and working around to the other end.

**For example**, at A the pressure is  $p_A$  which is equal to  $p_1$  and as we move to point (2) the pressure increases by  $\gamma_1 h_1$ . The pressure at  $p_2$  is equal to  $p_3$  and as we move upward to point (4) the pressure decreases by  $\gamma_2 h_2$ .

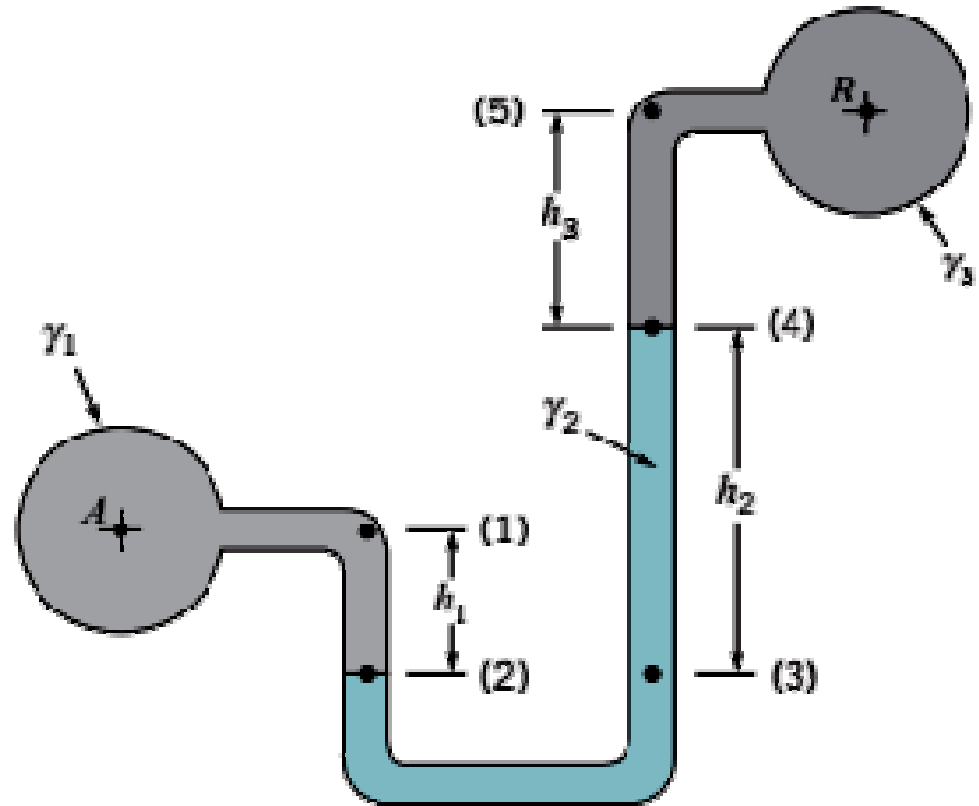


Fig: Differential U-tube manometer.

Similarly, as we continue to move upward from point (4) to (5) the pressure decreases by  $\gamma_3 h_3$ . Finally,  $p_5 = p_B$  since they are at equal elevations. Thus,

$$p_A + \gamma_1 h_1 - \gamma_2 h_2 - \gamma_3 h_3 = p_B$$

The pressure difference is

$$p_A - p_B = \gamma_2 h_2 + \gamma_3 h_3 - \gamma_1 h_1$$

**Example (2):** a U-tube gauge is arranged to measure the pressure difference between two points in a pipeline.

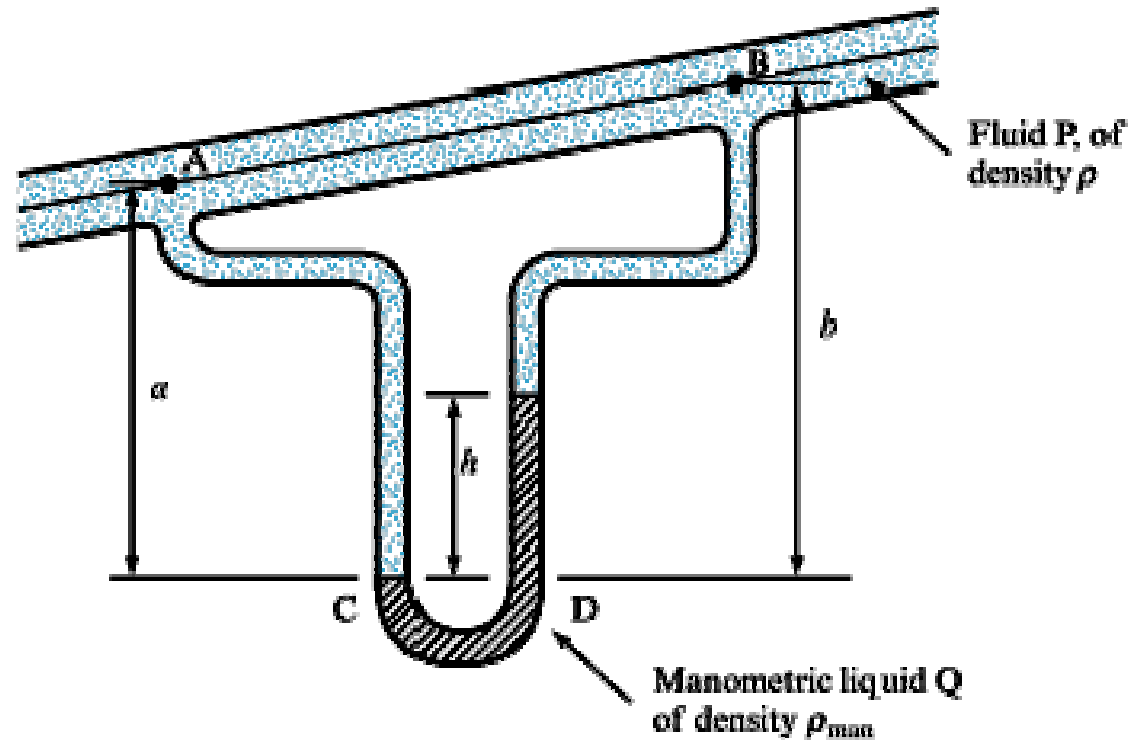


Fig: Measurement of pressure difference.

The principle involved in calculating the pressure difference is that the pressure at the same level  $CD$  in the two limbs must be the same, since the fluid in the bottom of the U-tube is at rest. For the left-hand limb,

$$p_C = p_A + \rho g a$$

For the right-hand limb,

$$p_D = p_B + \rho g(b - h) + \rho_{man} g h.$$

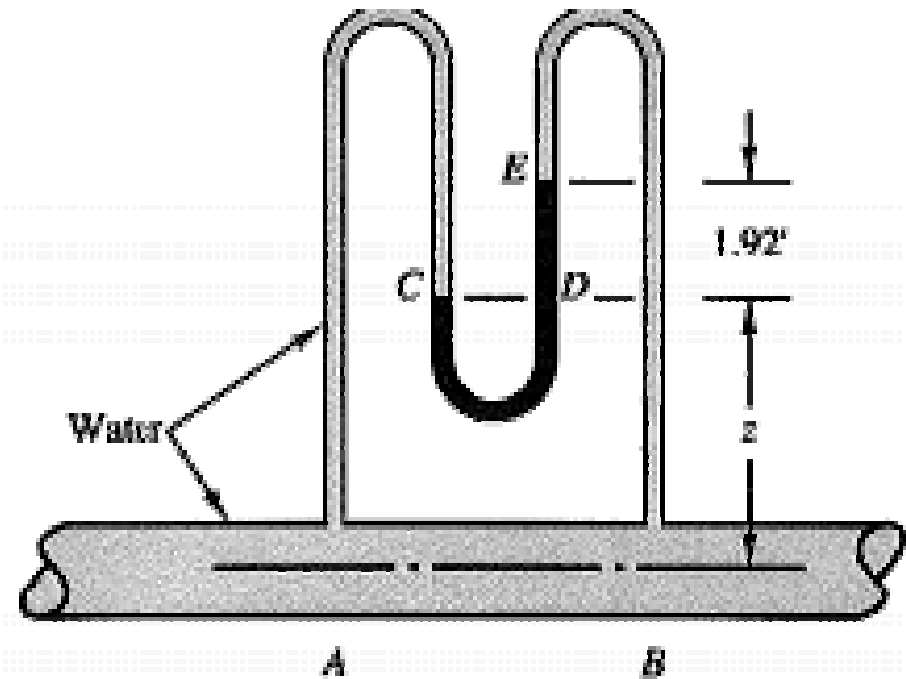
Since  $p_C = p_D$ ,

$$p_A + \rho g a = p_B + \rho g(b - h) + \rho_{man} g h$$

$$\text{Pressure difference} = p_A - p_B = \rho g(b - a) + hg(\rho_{man} - \rho)$$

### **Problem:**

A differential gage is attached to two cross sections  $A$  and  $B$  in a horizontal pipe in which water is flowing. The deflection of the mercury in the gage is 1.92 ft, the level nearer  $A$  being the lower one. Calculate the difference in pressure in psi between sections  $A$  and  $B$ .



**Solution:**

*Note:* A sketch will be of assistance in clarifying the analysis of all problems as well as in reducing mistakes. Even a single-line diagram will help.

pressure head at *C* = pressure head at *D*

or, using ft of water units,  $p_A/\gamma - z = [(p_B/\gamma) - (z + 1.92)] + (13.57)(1.92)$

Then  $p_A/\gamma - p_B/\gamma = \text{difference in pressure heads} = (1.92)(13.57 - 1) = 24.1 \text{ ft water}$

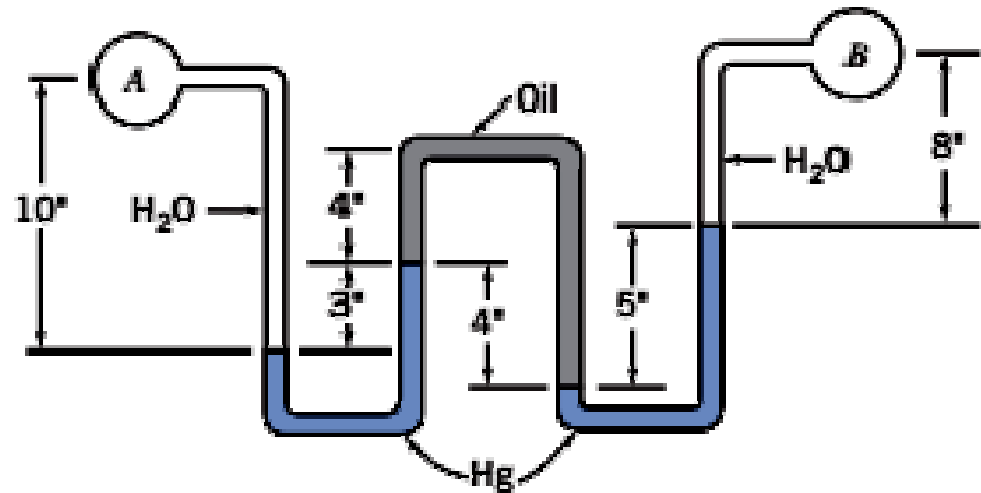
and  $p_A - p_B = (24.1)(62.4/144) = 10.44 \text{ psi.}$

If  $(p_A - p_B)$  had been negative, the proper interpretation of the sign would be that the pressure at *B* was greater than the pressure at *A* by 10.44 psi.

Differential gages should have the air extracted from all tubing before readings are taken.

### Problem:

Water flows through pipes A and B. Lubricating oil is in the upper portion of the inverted U. Mercury is in the bottom of the manometer bends. Determine the pressure difference,  $p_A - p_B$ , in units of  $\text{lbf/in.}^2$



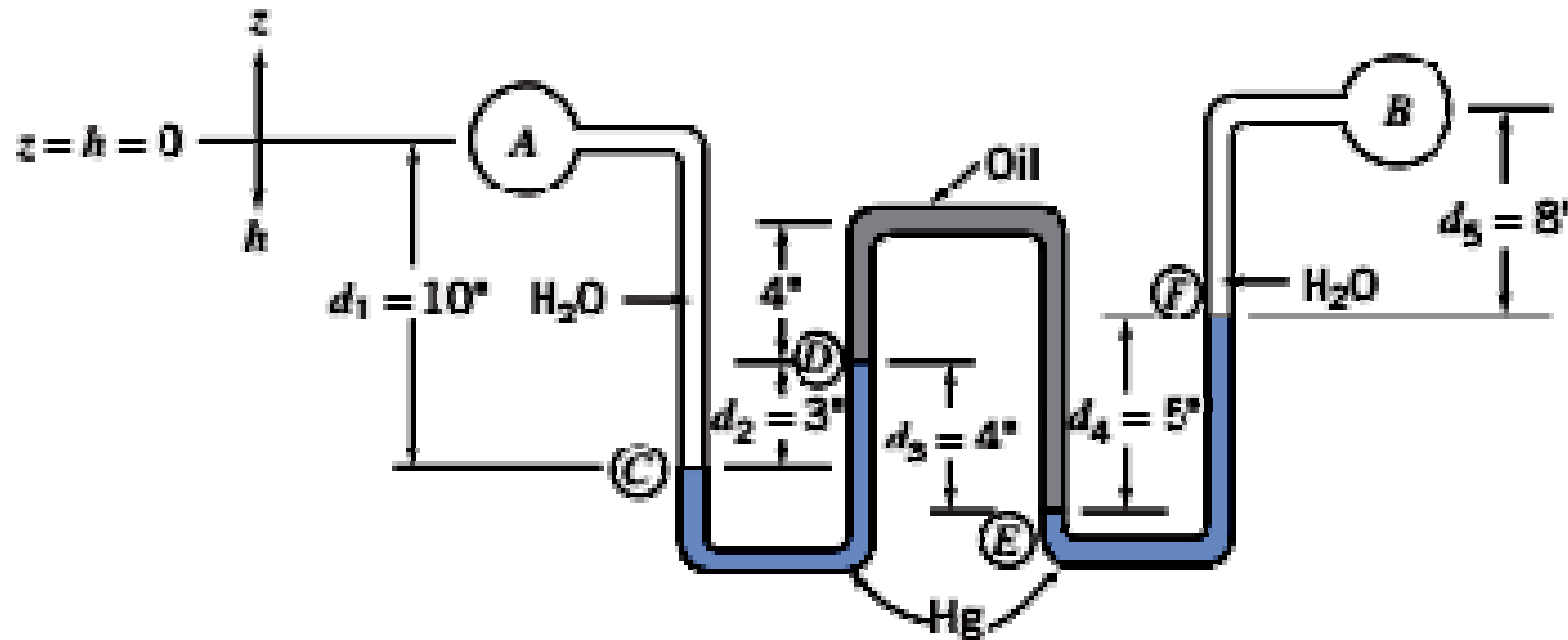
### Solution:

*Governing equations:*  $\Delta p = g \sum \rho h$        $SG = \frac{\rho}{\rho_{H_2O}}$

*Assumptions:* (1) Static fluid. (2) Incompressible fluid.

Applying the governing equation, working from point B to A

$$p_A - p_B = \Delta p = g(\rho_{H_2O}d_5 + \rho_{Hg}d_4 - \rho_{oil}d_3 + \rho_{Hg}d_2 - \rho_{H_2O}d_1)$$



Substituting  $\rho = SG\rho_{H_2O}$  with  $SG_{Hg} = 13.6$  and  $SG_{oil} = 0.88$ , yields

$$P_A - P_B = g(-\rho_{H_2O}d_1 + 13.6\rho_{H_2O}d_2 - 0.88\rho_{H_2O}d_3 + 13.6\rho_{H_2O}d_4 + \rho_{H_2O}d_5)$$

$$= g\rho_{H_2O}(-d_1 + 13.6d_2 - 0.88d_3 + 13.6d_4 + d_5)$$

$$P_A - P_B = g\rho_{H_2O}(-10 + 40.8 - 3.52 + 68 + 8) \text{ in.}$$

$$P_A - P_B = g\rho_{H_2O} \times 103.3 \text{ in.}$$

$$= 32.2 \frac{\text{ft}}{\text{s}^2} \times 1.94 \frac{\text{slug}}{\text{ft}^3} \times 103.3 \text{ in.} \times \frac{\text{ft}}{12 \text{ in.}} \times \frac{\text{ft}^2}{144 \text{ in.}^2} \times \frac{\text{lb} \cdot \text{s}^{-2}}{\text{slug} \cdot \text{ft}}$$

$$P_A - P_B = 3.73 \text{ lbf/in.}^2$$

Beginning at point *A* and applying the equation between successive points along the manometer gives

$$P_C - P_A = +\rho_{H_2O}gd_1$$

$$P_D - P_C = -\rho_{Hg}gd_2$$

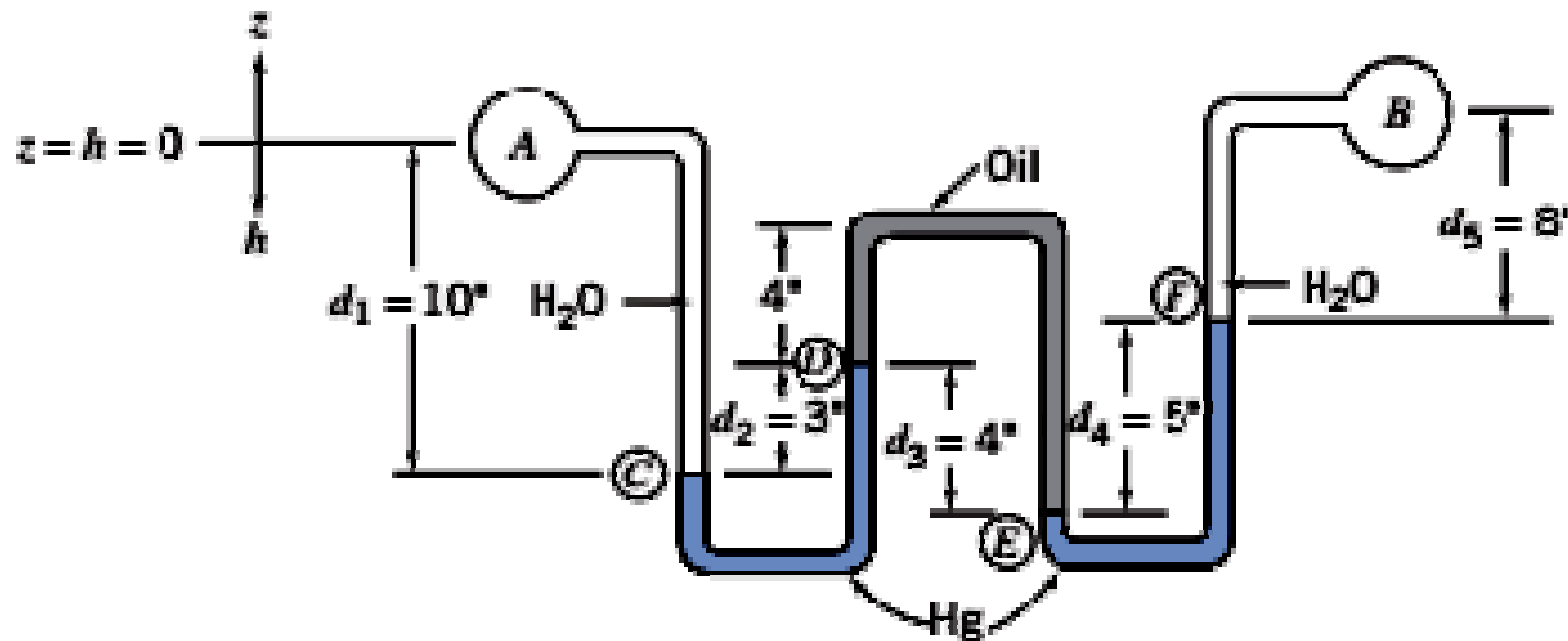
$$P_E - P_D = +\rho_{oil}gd_3$$

$$P_F - P_E = -\rho_{H_2O}gd_4$$

$$P_B - P_F = -\rho_{H_2O}gd_5$$

Multiplying each equation by minus one and adding, we obtain Eq. (1)

$$\begin{aligned} P_A - P_B &= (P_A - P_C) + (P_C - P_D) + (P_D - P_E) + (P_E - P_F) + (P_F - P_B) \\ &= -\rho_{H_2O}gd_1 + \rho_{Hg}gd_2 - \rho_{oil}gd_3 + \rho_{H_2O}gd_4 + \rho_{H_2O}gd_5 \end{aligned}$$



## Inclined-Tube Manometer

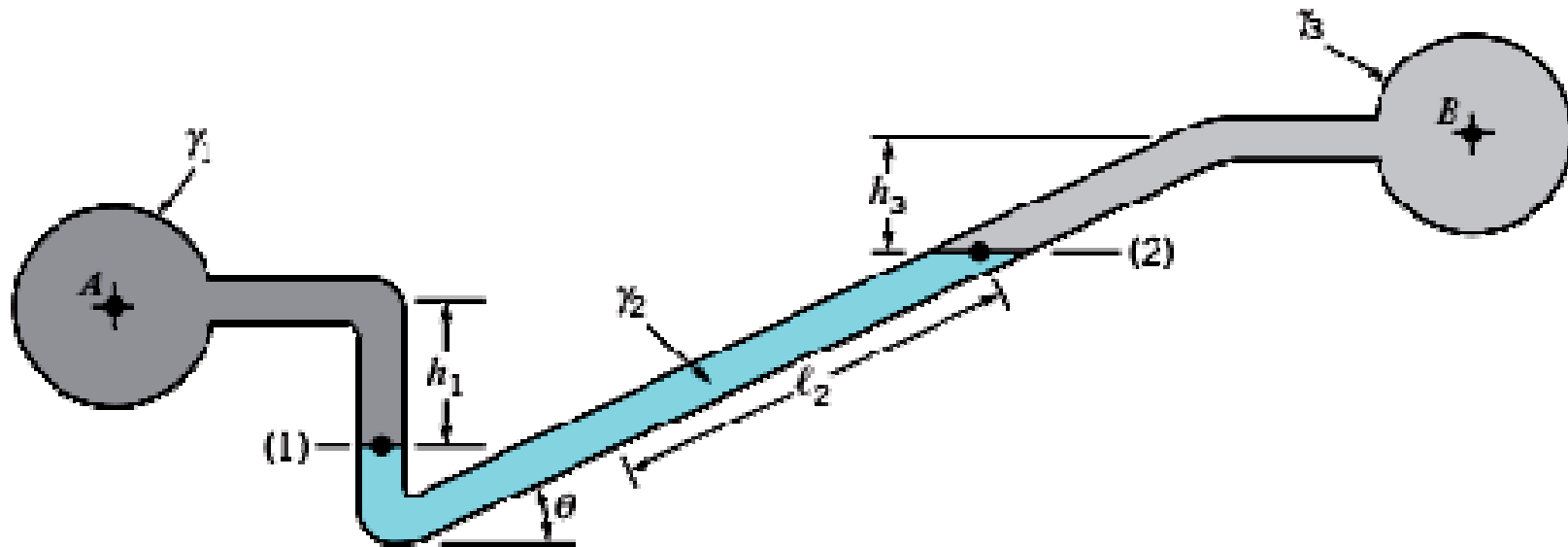
To measure small pressure changes, a manometer of the type shown in Fig. is frequently used. One leg of the manometer is inclined at an angle  $\theta$  and the differential reading  $l_2$  is measured along the inclined tube. The difference in pressure  $p_A - p_B$  can be expressed as

$$p_A + \gamma_1 h_1 - \gamma_2 l_2 \sin\theta - \gamma_3 h_3 = p_B$$

Or,

$$p_A - p_B = -\gamma_1 h_1 + \gamma_2 l_2 \sin\theta + \gamma_3 h_3$$

where the pressure difference between points (1) and (2) is due to the vertical distance between the points, which can be expressed as  $l_2 \sin\theta$ .



Thus, for relatively small angles the differential reading along the inclined tube can be made large even for small pressure differences. The inclined-tube manometer is often used to measure small differences in gas pressures so that if pipes  $A$  and  $B$  contain a gas then

$$p_A - p_B = \gamma_2 l_2 \sin \theta$$

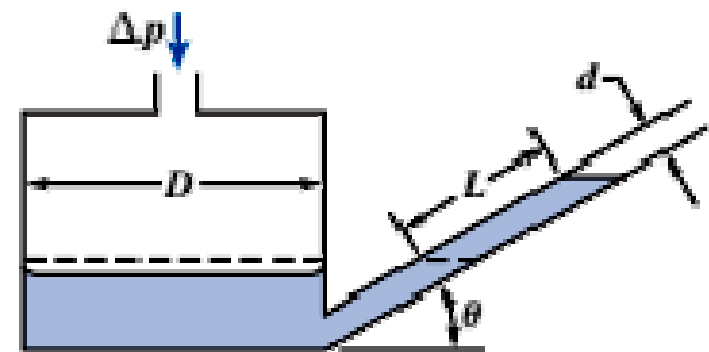
Or,

$$l_2 = \frac{p_A - p_B}{\gamma_2 \sin \theta}$$

where *the contributions of the gas columns  $h_1$  and  $h_3$  have been neglected.*

### Problem:

An inclined-tube reservoir manometer is constructed as shown. Derive a general expression for the liquid deflection,  $L$ , in the inclined tube, due to the applied pressure difference,  $\Delta p$ . Also obtain an expression for the manometer sensitivity, and discuss the effect on sensitivity of  $D$ ,  $d$ ,  $\theta$ , and  $SG$ .

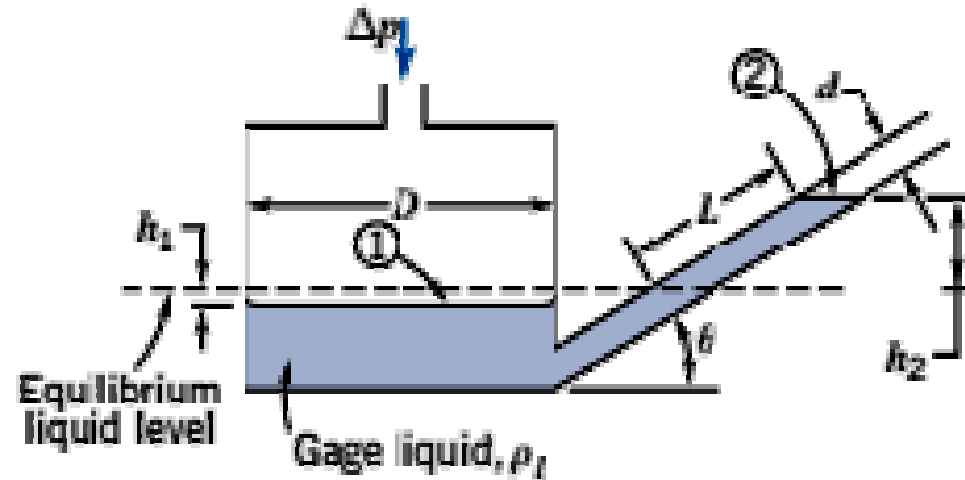


**Solution:**

Use the equilibrium liquid level as a reference.

**Governing equations:**

$$p - p_0 = \Delta p = \rho g h \quad SG = \frac{\rho}{\rho_{H_2O}}$$



**Assumptions:** (1) Static fluid. (2) Incompressible fluid.

Applying the governing equation between points 1 and 2

$$p_1 - p_2 = \Delta p = \rho_l g (h_1 + h_2) \quad \text{---(1)}$$

To eliminate  $h_1$ , we recognize that the volume of manometer liquid remains constant; the volume displaced from the reservoir must equal the volume that rises in the tube, so

$$\frac{\pi D^2}{4} h_1 = \frac{\pi d^2}{4} L \quad \text{or} \quad h_1 = L \left( \frac{d}{D} \right)^2$$

In addition, from the geometry of the manometer,  $h_2 = L \sin \theta$ .  
 Substituting into Eq. 1 gives

$$\Delta p = \rho g \left[ L \sin \theta + L \left( \frac{d}{D} \right)^2 \right] = \rho g L \left[ \sin \theta + \left( \frac{d}{D} \right)^2 \right]$$

Thus

$$L = \frac{\Delta p}{\rho g \left[ \sin \theta + \left( \frac{d}{D} \right)^2 \right]}$$

To find the sensitivity of the manometer, we need to *compare this to the deflection  $h$  a simple U-tube manometer, using water (density  $\rho$ ), would experience,*

$$h = \frac{\Delta p}{\rho g}$$

The sensitivity,  $s$  is then

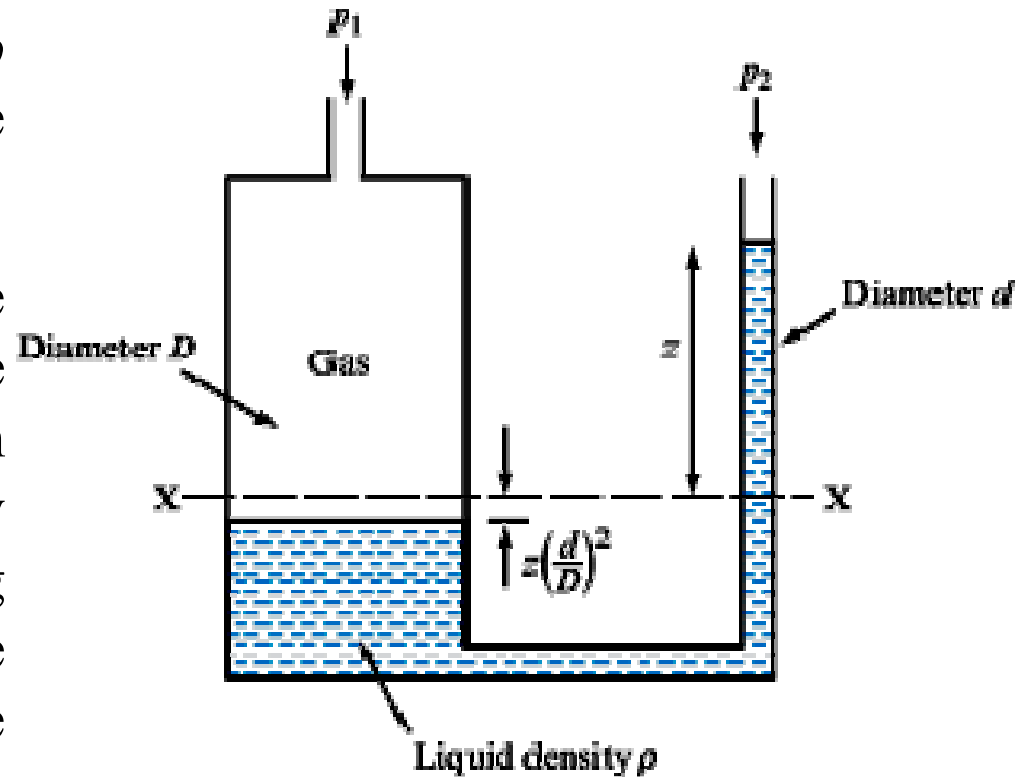
$$s = \frac{L}{h} = \frac{1}{SG_1 \left[ \sin \theta + \left( \frac{d}{D} \right)^2 \right]}$$

where we have used  $SG_1 = \rho_l/\rho$ . *This result shows that to increase sensitivity,  $SG_1$ ,  $\sin \theta$ , and  $d/D$  each should be made as small as possible.*

## U-tube with one leg enlarged

If the fluid P is a gas its density  $\rho$  can usually be treated as negligible compared with  $\rho_{\text{man}}$ .

Industrially, the simple U-tube manometer has the disadvantage that the movement of the liquid in both limbs must be read. By making the diameter of one leg very large as compared with the other, it is possible to make the movement in the large leg very small, so that it is only necessary to read the movement of the liquid in the narrow leg.



Assuming that the manometer in Fig. is used to measure the pressure difference  $p_1 - p_2$  in a gas of negligible density and that XX is the level of the liquid surface when the pressure difference is zero, then, when pressure is applied, the level in the right-hand limb will rise a distance  $z$  vertically.

Volume of liquid transferred from left-hand leg to right-hand leg  
 $= z \times (\pi/4)d^2$

Fall in level of the left-hand leg

$$= \frac{\text{Volume transferred}}{\text{Area of left-hand leg}} = \frac{z \times (\pi/4)d^2}{(\pi/4)D^2} = z \left( \frac{d}{D} \right)^2.$$

The pressure difference,  $p_1 - p_2$ , is represented by the height of the manometric liquid corresponding to the new difference of level:

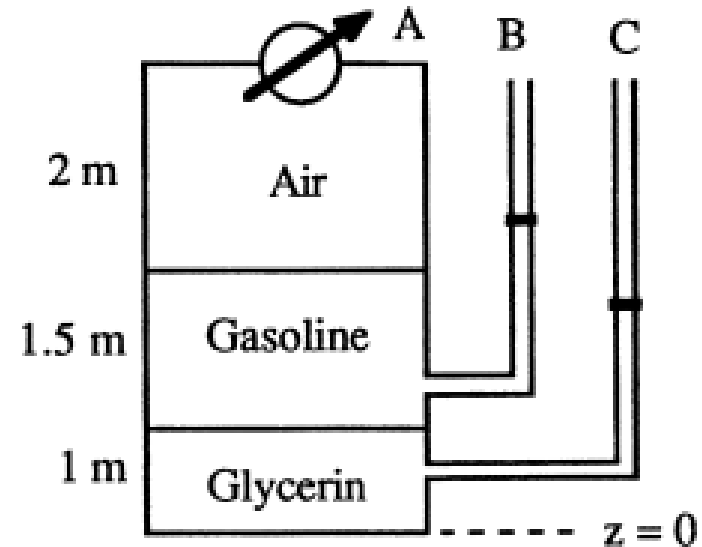
$$p_1 - p_2 = \rho g [z + z(d/D)^2] = \rho g z [1 + (d/D)^2],$$

or, **if D is large compared with d,**

$$p_1 - p_2 = \rho g z$$

**Problem:**

In Fig., sensor A reads 1.5 kPa (gage). All fluids are at 20°C. Determine the elevations  $z$  in meters of the liquid levels in the open piezometer tubes B and C.



**Solution:**

The specific weights are

$$\gamma_{\text{air}} \approx 12.0 \text{ N/m}^3, \gamma_{\text{gasoline}} = 6670 \text{ N/m}^3, \text{ and } \gamma_{\text{glycerin}} = 12360 \text{ N/m}^3$$

*Then apply the hydrostatic formula from point A to point B:*

$$1500 \text{ N/m}^2 + (12.0 \text{ N/m}^3)(2.0 \text{ m}) + 6670 \text{ N/m}^3(1.5 + 1 - z_B)\text{m} = p_B = 0 \text{ (gage)}$$

$$z_B = 2.73 \text{ m} \quad \text{(23 cm above the gasoline-air interface) Ans.}$$

*Then apply the hydrostatic formula from point A to point C:*

$$1500 + 12.0(2.0) + 6670(1.5) + 12360(1.0 - z_C) = p_C = 0 \text{ (gage)} \quad (\text{N/m}^2)$$

$$z_C = 1.93 \text{ m} \quad \text{(93 cm above the gasoline-glycerin interface) Ans.}$$